PC 20 3.3 - *Completing the Square*

Changing Forms

A quadratic function can be written in several different forms. In the previous lesson we studied **standard form**, which is written as $y = ax^2 + bx + c$. We also have studied **vertex form**:

$y = a(x - p)^2 + q$.

Changing ***Vertex Graphing Form to Standard form is easy***… just ***expand*** and ***gather*** like terms

$y = (x - 2)^2 + 3$

$y = (x - 2)(x - 2) + 3$

$y = x^2 - 4x + 4 + 3$

$y = x^2 - 4x + 7$

*NOTICE that if we start with STANDARD FORM, the process to change to VERTEX FORM is not as easy… we will learn how to do that today

Completing the Square

Vertex form is especially useful because it shows the **vertex** of the parabola in the form itself. The vertex is the point (p, q). Because ***we can't easily see the vertex in standard form***, we need a process that changes forms.

The process of changing a quadratic from standard form to vertex form is called **completing the square**. This process is based on "*creating*" a ***perfect square trinomial***.

Why Complete the Square?

Notice above, the reverse process of expanding is factoring…

$y = x^2 - 4x + 7$

$y = x^2 - 4x + 4 + 3$

$y = (x - 2)(x - 2) + 3$

$y = (x - 2)^2 + 3$

we need to take at least a portion of the standard form and write it as a factor of degree 2 (squared).

If we had a STANDARD form equation that ***WAS A PERFECT SQUARE***, then rewriting IN VERTEX FORM would be easy!!

Ex. $y = x^2 - 6x + 9$ this can be rewritten as

$y = (x - 3)(x - 3)$ and in condensed form

$y = (x - 3)^2$ This is *STANDARD FORM*

Now consider a similar function; $y = x^2 - 6x + 5$ This is very close to a perfect square. Obviously the 5 doesn't fit. What we need to do is CHANGE THE CONSTANT to make a perfect square component. But we can't just change the 5 to a 9… We can rewrite it by *completing the square*.

First, let's make room for the PROPER constant for the ***x² − 6x*** part

y = (x² − 6x + _____*) + 5*

Note: this constant will always be added due to the 'squared' nature of the factor

But we also need to balance the constant we put into the equation. So whatever we add, we must also take away to make a NET ZERO for the equation. Write a second space for this ***outside the brackets***.

y = (x² − 6x + _____*) + 5* **-** _____

To determine the correct constant to add, we look at the **coefficient of x**. *It is* ***−6***

Take half of −6 →−3. **Then square this number**: (−3)² = 9. The ***number 9*** is the number needed to create a perfect square.

Because we need to net zero for our adjustments, we ***also subtract*** 9

y = (x² − 6x + __9__*) + 5 -* ___9___

REMEMBER: **half the x term coefficient, then square it**

The first three terms NOW form a **perfect square trinomial**: ***(x² − 6x + 9) → (x − 3) (x − 3) → (x − 3)²***

So… y = (x² − 6x + 9) + 5 − 9 BECOMES y = (x − 3)² + 5 − 9

Combining the remaining constants we get: ***y = (x − 3)² − 4***. This is vertex form, so the ***vertex is (3, −4)***

The General Process

When completing the square, follow a consistent sequence. Suppose we begin with ***y = x² - 8x + 3***

Step 1: Make space to add the new constant, and make space to also subtract it.

Example: ***y = x² − 8x + 3***

becomes ***y = (x² − 8x +*** ___***) + 3 -*** ____

Step 2: Take half of the x coefficient. The coefficient of *x* is *−8*, so half of *−8* is ***−4***

Step 3: Square the result. *(−4)² = 16*. **This** is the number needed to complete the square (it will always be positive)

Step 4: Place this number in your spaces

y = (x² − 8x + ***_16__) + 3 -*** ***_16__***

Step 5: Factor the perfect square trinomial part and re-write that section. Combine constants

y = (x – 4)² - 13 → The ***vertex is (4, −13)***

What Happens When a ≠ 1?

We need to add *two extra steps* when the coefficient of x^2 is not 1.

Consider ***y = 5x² − 30x + 7***

First, factor 5 from the first two terms so the x^2 term coefficient is 1:

y = 5(x² − 6x) + 7

Now complete the square ***inside the brackets*** like we did above.

y = 5(x² − 6x + _____) + 7

Half of −6 is −3, and $(-3)^2 = 9$. Add 9 inside the brackets:

y = 5(x² − 6x + 9) + 7

****Here is where we have to pay close attention!** We also have to ***subtract what we added***… but ***what did we add?*** NOTICE WE ADDED 5 TIMES 9… OR 45… so this is what we subtract outside the brackets…

y = 5(x² − 6x + 9) + 7 → ***y = 5(x² − 6x + 9) + 7 – 45***

The simplified version becomes *y = 5(x − 3)² − 38*

The *vertex is* ***(3, −38)***, the ***'a'*** value is STILL 5… as it was in the Standard Form.

Practice – Completing the square:

A. ***y = x² − 6x + 7***

B. ***y = 5x² − 10x + 7***

Sometimes you will have to work with fractions

C. ***y = 4x² − 28x − 23***

Completing the Square:

A Quick Checklist

1. Factor out the coefficient of x^2 from the first two variable terms (if necessary)
2. Group the x^2 and x terms.
3. Take half of the coefficient of x.
4. Square that number.
5. Add (inside brackets) and subtract (outside brackets) the same number.
6. Factor the resulting perfect square trinomial.
7. Simplify.
8. Identify the vertex.

Word Problem:

Suppose you are advising a business on their production costs. The production costs can be modelled by the function $C(n) = 0.3n^2 – 48.6n + 13500$. Where C is the Cost, and n is the number of units produced in thousands.

a.) Identify the vertex by completing the square
b.) identify what the vertex means for the company.

Solution: Use the steps to complete the square. Use your calculator and work with decimals to two or three decimal places and answer the questions.

Connection between 'p' and completing the square.

The x-value of the vertex is 'b/2… but also divided by 'a'… and also the negative which is exactly $\boldsymbol{x = \frac{-b}{2a}}$

Homework: