3.1 Investigating Quadratic Functions in Vertex Form

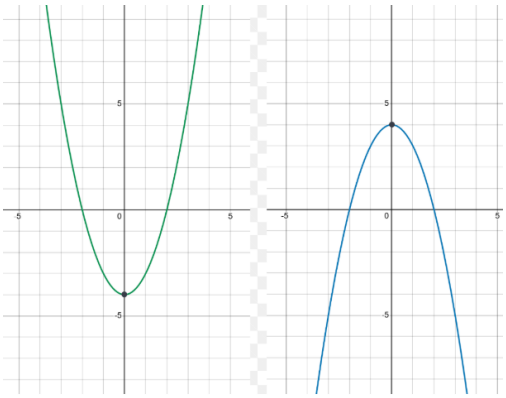
Part – 1 Quadratic Functions

A **quadratic function** is a function in which the *degree is 2*. The **degree** is the highest exponent of the variable.

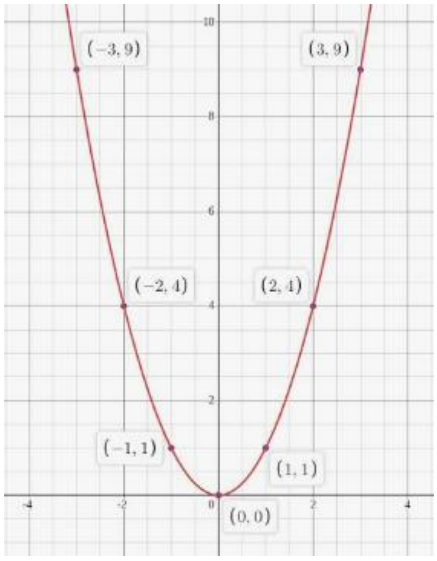
A quadratic function can be represented by a graph called a **parabola**.

Depending on the equation, a parabola can open

"up like a cup" **U** or ***"down like a frown"*** ∩



The simplest quadratic function is: $y = x^2$



Its graph provides a useful starting point for understanding all other quadratic functions.

Graphing ***(x,y)***: notice the points are ***(x, x^2)***

The Vertex

The **vertex** is highest or lowest *point* on the parabola.

It contains the *maximum or minimum y value* for the function.

For a parabola that opens **upward**, the vertex is the **minimum point**, because it is the lowest point on the graph.

For a parabola that opens **downward**, the vertex is the **maximum point**, because it is the highest point on the graph.

It is also the *turning point* of a parabola (the point where the parabola changes direction).

The graph of $f(x) = x^2$ has its lowest point at (0, 0), so therefore it is the vertex.

Vertex Form

A quadratic equation written in vertex form includes ***parameters*** that indicate the ***vertex***, *and the* ***direction*** *and* ***width*** *of opening*: $f(x) = a(x - p)^2 + q$

The ***vertex is (p, q)***, and the 'a' value indicates the 'width' of the parabola ('normal' width is ***a=1***)

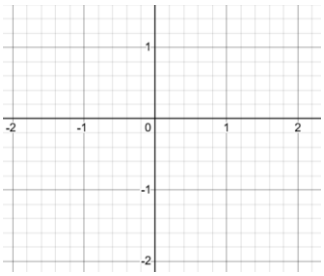
Note that $f(x) = a(x - p)^2 + q$ ***is the same as*** $y = a(x - p)^2 + q$

Notes and Questions:

For the function $f(x) = x^2$, the coordinates of the vertex are (0,0) and width is normal, so:

$f(x) = a(x - p)^2 + q$ *becomes* $f(x) = 1(x - 0)^2 + 0$

when simplified, this is the same as $f(x) = x^2$

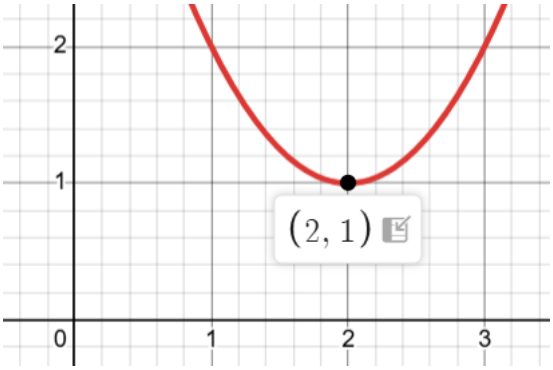


Q: What does the function $y = x^2 - 2$ look like?

In vertex form, it is $y = 1(x - 0)^2 + (-2)$ So the vertex *(0,-2)*

Another example, $y = (x - 2)^2 + 1$ has vertex (2, 1).

Notice that the equation contains (x − 2), so the x-coordinate of the vertex is positive 2.



Understanding the Parameter '*a*'

In vertex form, $f(x) = a(x - p)^2 + q$, the **parameter *a*** controls the ***direction*** and ***width*** of the parabola.

If $a > 0$, the parabola opens **upward**.

If $a < 0$, the parabola opens **downward**.

If $|a| > 1$, or the ***'number is larger than one'***, the parabola is **narrower** than $y = x^2$.

If $0 < |a| < 1$, or the ***'number is smaller than one'***, the parabola is **wider** than $y = x^2$.

If $a = 1$, the graph has a ***'normal width'*** and is the same width as $y = x^2$.

Axis of Symmetry

For a quadratic in vertex form, $f(x) = a(x - p)^2 + q$, the axis of symmetry is the vertical line that divides the parabola into two equal halves. This equation is $x = p$.

For example, if $f(x) = (x - 2)^2 + 1$ *as in the example above, the axis of symmetry (a.o.s.) is* $x=2$

What is the a.o.s. for $f(x) = 2(x - 3)^2 + 4$? $x =$ _____

The axis of symmetry passes through the vertex.

Notes and Questions:

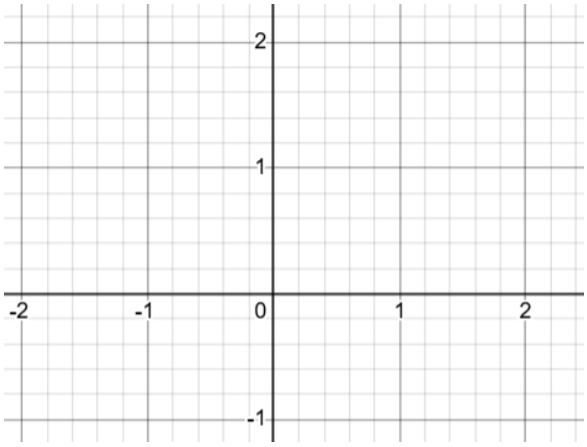
Maximum and Minimum Points

The direction in which a parabola opens determines whether its vertex is a maximum or minimum.

If ***a > 0***, the parabola opens upward and the vertex is a **minimum point**.

If ***a < 0***, the parabola opens downward and the vertex is a **maximum point**.

For example,***y = 2(x − 1)² + 1*** opens upward, (a=positive) so (1, 1) is its minimum point.



y = −2(x − 1)² + 1 opens downward (a = negative), so (1, 1) is its maximum point.

A negative value of 'a' reflects the parabola in the x-axis.

Comparing ***y = x²*** and ***y = −x²***

All the y values (created by squaring x) are multiplied by -1 so the parabola flips under the x-axis.

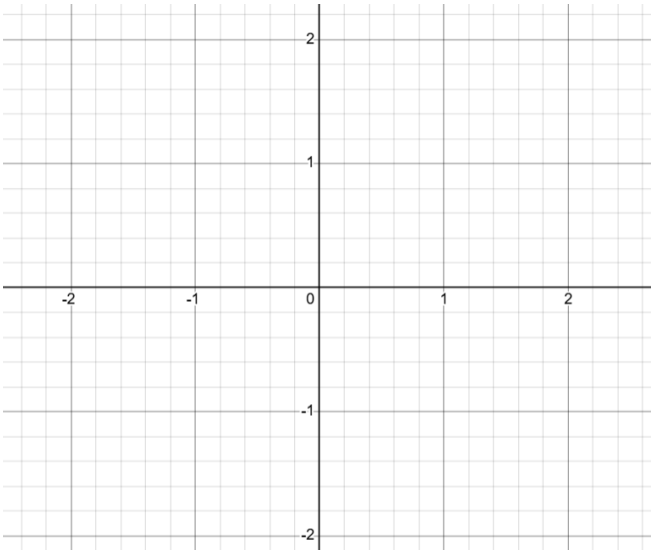
'a' summary:

https://www.desmos.com/calculator/c3cde09be4

y = −x² opens downward

y = 3x² is narrower than y = x²,

y = 0.5x² is wider than y = x²



Notes and Questions:

Part 2- Transformations of the Basic Parabola https://www.desmos.com/calculator/c3cde09be4

We can view ***a, p,*** and ***q*** not only as the *width parameter* and *coordinates for the vertex* of a quadratic function, but also the **parameters that transform the basic quadratic function of *f(x)=x²*** The values of a, p, and q determine how the graph is transformed from the basic graph $y = x^2$.

$$f(x) = a\,(x - p)^2 + q$$

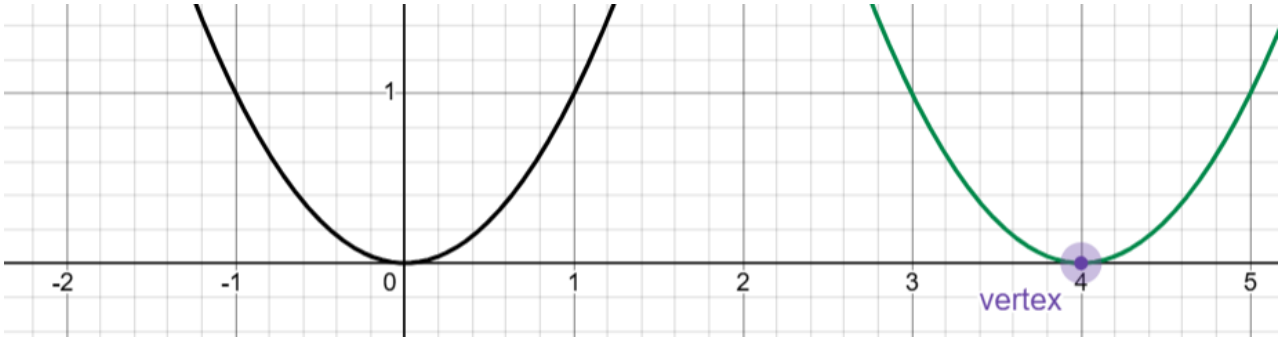
Horizontal Translation

The **parameter *p*** determines the horizontal position of the parabola.

If **p > 0**, the entire graph ***moves p units to the right.***

If **p < 0**, the entire graph ***moves p units to the left***.

For example, $f(x) = (x - 4)^2$ moves the basic parabola 4 units to the **right**.



Another example: $f(x) = (x + 2)^2$ can be rewritten as $(x - (-2))^2$, so it moves 2 units to the **left**.

Vertical Translation

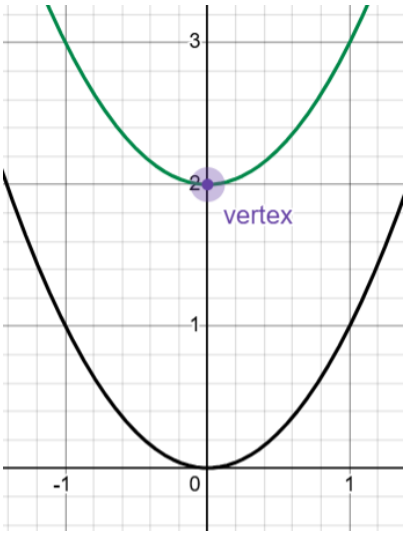
The **parameter q** determines the vertical position of the parabola.

If **q > 0**, the graph moves q units **upward**.

If **q < 0**, the graph moves |q| units **downward**.

Example: $f(x) = x^2 + 2$ moves the basic parabola 2 units upward.

Example: $f(x) = x^2 - 3$ moves the basic parabola 3 units downward.



Reading the Complete Vertex Form

Consider: $y = -2(x - 3)^2 + 5$

We can identify each important feature directly from the equation:

$a = -2 \quad p = 3 \quad q = 5$

Vertex: (3, 5)

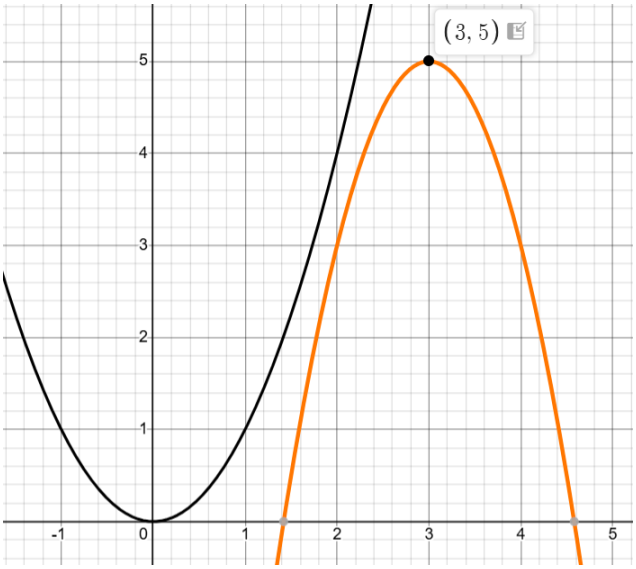
Axis of symmetry: x = 3

Opens downward because *a* is negative

Has a maximum point at (3, 5)

Is narrower than $y = x^2$ because $|a| > 1$

Moves 3 units right and 5 units up from $y = x^2$



Note: Using ***'a'*** to sketch other points on the graph

- From the vertex, move 1 unit to the left or right and 'a' units in the vertical direction to get two more points on the parabola.
- Move 2 units left or right and then move 4a units vertically
- Move 3 units left or right and then move 9a units vertically, etc, etc

Finding an Equation from the Vertex and Another Point

Sometimes you are given the vertex and one other point on the parabola and must determine the equation. Start with $f(x) = a\,(x - p)^2 + q$

Suppose the vertex is (2, 1) and the parabola passes through (4, 9). Substitute the vertex first:

$f(x) = a(x - \mathbf{2})^2 + \mathbf{1}$

we must calculate for ***'a'*** next:

Substitute the given point *(4, 9) in the x and f(x)* positions*:*

$\mathbf{9} = a(\mathbf{4} - 2)^2 + 1$

$9 = 4a + 1$

$8 = 4a$

$a = 2$

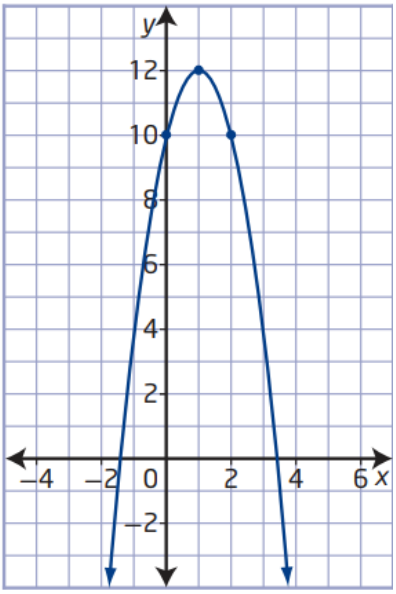
Therefore: $f(x) = 2(x - 2)^2 + 1$

Your Turn: Suppose the vertex is (-5, 3) and the parabola passes through (-3, 15). Find the Equation in vertex form.

Finding an Equation from the Graph

Find the Equation from the information on the graph:

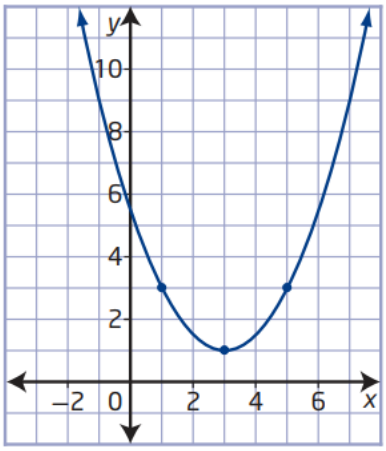
Eg.1



We can see that the vertex is (1,12).
The other points on the graph are 1 unit to the left / right of the vertex. We know that the points on the graph should be 'a' units from this point.
So, p=1, q=12, a=-2

Therefore the equation is: $f(x) = -2(x - 1)^2 + 12$

Your Turn: What is the equation of the parabola below. Be careful… the points given are 2 units from the vertex… check your notes to help you find 'a'.



Homework: