

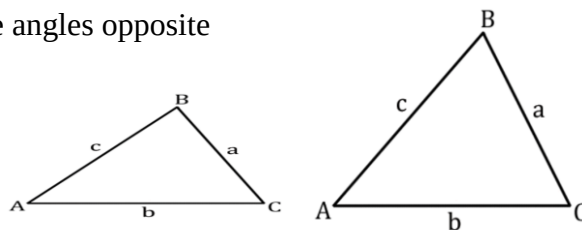
PC 20 2.3 - SINE LAW AND THE AMBIGUOUS CASE

The Sine Law

The **sine law** is a relationship that can be used to find unknown sides or angles in a triangle. It is especially useful when you do not have enough information to use the Pythagorean theorem, the cosine law, or the triangle is not a right triangle.

In triangle ABC, the sides are named according to the angles opposite them:

- side a is opposite angle A
- side b is opposite angle B
- side c is opposite angle C



The sine law is built on the connection that as an angle increases so does the side opposite of it, and this also changes the other angles and side lengths. It can be written in either of the following equivalent forms:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{OR} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The important idea is that **each side must be paired with the opposite angle**. For example, side a must always be paired with angle A .

When Can You Use the Sine Law?

The sine law can be used when you **know an angle and its opposite side length**

Solving for an Unknown Side

Eg. Suppose you know $A = 40^\circ$, $B = 65^\circ$, and $a = 8$ cm.

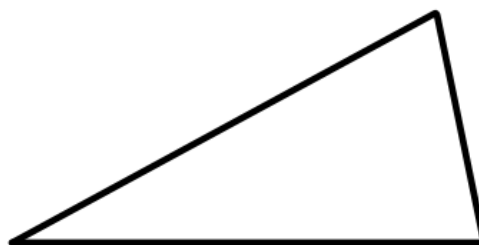
First find the third angle:

$$C = 180^\circ - 40^\circ - 65^\circ$$

$$C = 75^\circ$$

Now use the sine law to find b :

$$\frac{8}{\sin 40^\circ} = \frac{b}{\sin 65^\circ}$$



$$\frac{a}{\sin A} = \frac{b}{\sin B}$$



When using a calculator to find an *angle*, make sure it is in **degree mode** when the angles are given in degrees. This is when you use the 2nd function to find the angle. Here is an example

Solving for an Unknown Angle

Eg. Suppose you know $a = 7$ cm, $b = 10$ cm, and $A = 30^\circ$.

Using the sine law: $\frac{\sin 30^\circ}{7} = \frac{\sin B^\circ}{10}$

(when solving for an angle, it is often easier to use the second variation of the sine law where the angles are on top of the fraction)

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

*To find B, use the inverse sine function

However, this is where we must be careful. *The sine function can produce the same value for an acute angle and an obtuse angle.* More on that later.

Understanding the Ambiguous Case

The **ambiguous case** occurs when the given information consists of two sides and an angle opposite one of the given sides (SSA).

The situation is called “ambiguous” because the information may describe:

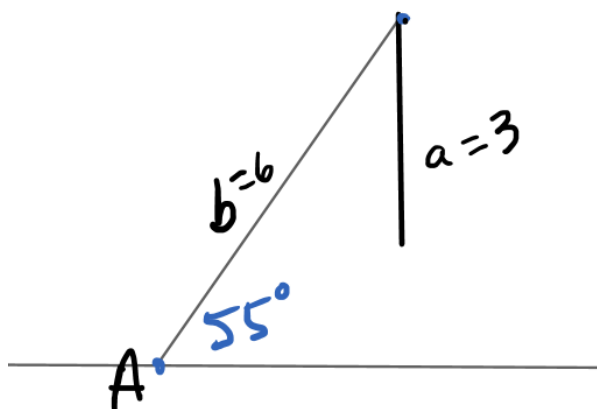
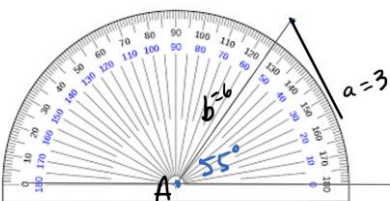
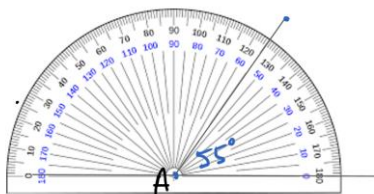
- no triangle
- exactly one triangle
- or two different triangles

Example: Can you sketch a triangle with the given dimensions?

Angle A = 55 degrees

Side a = 3cm

Side b = 6cm



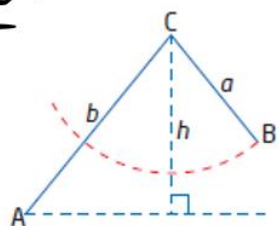
The key is the ***h*** value (***height***) of the triangle



The **height** is the MINIMUM length that 'a' must be to form a triangle.

This minimum height forms a RIGHT triangle., therefore the $h = b \sin A$

Case 1



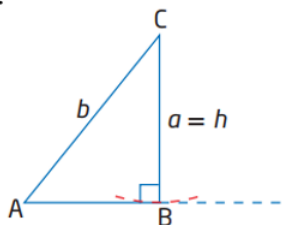
$a < h$
 $a < b \sin A$
 no solution

Recall that
 $h = b \sin A$.

Case 1: If the given 'a' value is LESS than h , then NO triangle can be formed.

Why is there no solution in this case?

Case 2



$a = h$
 $a = b \sin A$
 one solution

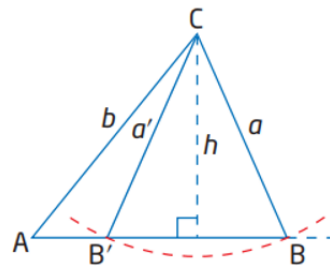
Case 2: If the given 'a' value is EQUAL TO h , then ONE triangle can be formed.

Why Can Two Triangles Occur?

Imagine one side of a triangle is fixed and the opposite side is able to swing around a fixed point. Depending on the lengths involved, the moving side may reach the fixed endpoint in two different positions.

This can create two different triangles that have the same given information – sketch both triangles below

Case 3



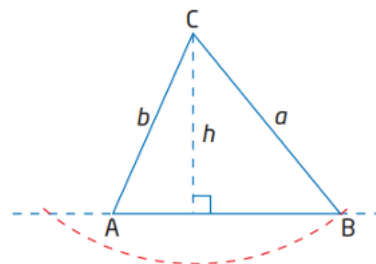
$h < a < b$
 $b \sin A < a < b$
 two solutions



Finally, in Case 4:

If the opposite side 'a' is greater than the adjacent side 'b', then there is **ONE TRIANGLE** formed.

Case 4



$a \geq b$
one solution

In Summary

When A is acute:

- If $a < h$, there is **no solution**.
- If $a = h$, there is **one solution**.
- If $h < a < b$, there are **two solutions**.
- If $a \geq b$, there is **one solution**

**The ambiguous case causes 2 possible triangles when $h < a < b$*

Example of the Height Test

Example: Given

$$\begin{aligned} A &= 30^\circ \\ a &= 6 \text{ cm} \\ b &= 10 \text{ cm} \end{aligned}$$

Determine how many triangles can be formed, and make a rough sketch of both given the information in the question.

Solution: First Step-Sketch the diagram. Then, calculate the minimum length required for the opposite side ($h = b \sin A$):

$$\begin{aligned} h &= b \sin A \\ h &= 10 \sin 30^\circ \\ h &= 5 \text{ cm} \end{aligned}$$

Now compare this with the given 'a' value and 'b' value:

$$h < a < b$$

$$5 < 6 < 10$$

Therefore, there are **two possible triangles**. Sketch...

$a < h$	No possible triangle
$a = h$	One possible right-angle triangle
$h < a < b$	The ambiguous case! There are 2 possible triangles
$a \geq b$	One possible triangle



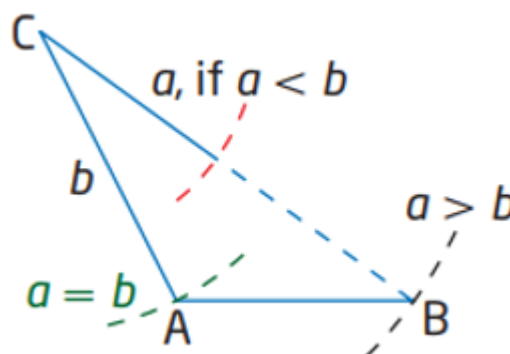
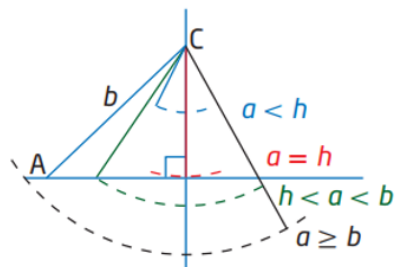
The ambiguous case behaves differently when the given angle is **obtuse**.

If A is obtuse:

- If $a \leq b$, there is **no solution**.
- If $a > b$, there is **one solution**.

This makes sense because *the side opposite an obtuse angle must be the longest side* of the triangle.

In Summary:



Practice:

1. Can you make a triangle with the given dimensions? How many??
Angle A = 38 degrees
Side a = 5cm
Side b = 8cm
2. Can you make a triangle with the given dimensions? How many??
Angle A = 38 degrees
Side a = 10cm
Side b = 8cm
3. Can you make *triangle PQR* with the given dimensions? Give all dimensions for angles and sides for all triangles formed. (This is called “*solving*” the triangles)
Angle Q = 43 degrees
Side q = 15cm
Side p = 20cm

Homework:

