Summary of the Use of Justifications on the AP Calculus Exam (Please note that these have changed over the years and answer keys from former years may reflect different expectations than this year)

Students are expected to demonstrate their knowledge of calculus concepts in 4 ways.

1. Numerically (Tables/Data)
2. Graphically
3. Analytically (Algebraic equations)
4. Verbally

The verbal component occurs often on the free response portion of the exam and requires students to explain and/or justify their answers and work. It is important that students understand what responses are valid for their explanations and justifications.

How do you know that a written answer is expected? Look for phrases such as:

- *Justify you answer*
- *Explain your reasoning*
- *Why?*
- *Why not?*
- *Give a reason for your answer*
- *Explain the meaning of a definite integral in the context of the problem.*
- *Explain the meaning of a derivative in the context of the problem.*
- *Explain why an approximation overestimates or underestimates the actual value*

How do you answer such a question? The short answer is to determine which theorem or definition applies and then show that the given situation specifically meets (or fails to meet) the hypotheses of the theorem or definition.

General Tips and Strategies for Justifications

1. A quality explanation does not need to be too wordy or lengthy. A proper explanation is usually very precise and short. Once a statement is made, STOP WRITING!!! Too often, students give a correct explanation, but continue to further elaborate and end up contradicting themselves or making an incorrect assertion which forfeits any points they could have earned.
2. Students commonly mix ideas in their explanations which cause them to not earn points. For example: "a function $f(x)$ is increasing" is equivalent to writing "$f'(x) > 0$". However, students often write "$f'(x)$ is increasing" when they intended to write "$f'(x) > 0$".
3. Avoid using pronouns in descriptions. Be specific! Do not write statements that begin with "The function…", "It…", or "The graph…". These are too general and the reader will not assume which function or graph is referenced. Name the functions by starting your statement with the phrase "$f(x)$ …" or "$f'(x)$ …", etc.
4. Know and understand proper mathematical reasons for the ideas covered in this session. Use the precise wording offered today and be assured that these are mathematically correct justifications that will earn points.
5. Make sure to show that the necessary conditions are met BEFORE using theorems like the Mean Value Theorem, Intermediate Value Theorem, Continuity, etc… L'Hopital's rule, etc…

- *To show that a* ***theorem applies*** *state and show that all its hypotheses are met. To show that a* ***theorem does not apply*** *show that at least one of the hypotheses is not true (be specific as to which one).*

Here are several concepts that have required explanations and justifications on free response questions over the past several years.

1. Riemann Sums as an over/under approximation of area
2. Relative minimums/maximums of a function
3. Points of inflection on a function
4. Continuity of a function
5. Speed of a particle increasing/decreasing
6. Meaning of a definite integral in context of a problem
7. Absolute minimum/maximum of a function
8. Using Mean Value Theorem
9. Intervals when a function is increasing/decreasing (particle motion)
10. Tangent lines as an over/under approximation to a point on a function
11. Use of L'Hopital's rule to find limits

A short version of common things to establish before you can come to a conclusion:

- ***Increasing, decreasing, local extreme values,*** *and* ***concavity*** *are all justified by reference to the function's derivative. The table below shows what is required for the justifications. The items in the second column must be given (perhaps on a graph of the derivative) or must have been established by the student's work.*

In order to Conclude that:	*You must clearly establish that:*
y is increasing	*y' > 0 (above the x-axis)*
y is decreasing	*y' < 0 (below the x-axis)*
y has a local minimum	*y' changes – to + (crosses x-axis below to above)* **or** $y' = 0$ and $y'' > 0$
y has a local maximum	*y' changes + to – (crosses x-axis above to below)* **or** $y' = 0$ and $y'' < 0$
y is concave up	*y' increasing (going up to the right)* **or** $y'' > 0$
y is concave down	*y' decreasing (going down to the right)* **or** $y'' < 0$
y has point of inflection	*y' extreme value (high or low points)* **or** y'' *changes sign.*

SOME SPECIFICS THAT ARE A BIT MORE IN DEPTH THAN ABOVE:

Continuity

A function is continuous on an interval if it is continuous at every point of the interval. Intuitively, a function is continuous if its graph can be drawn without ever needing to pick up the pencil. This means that the graph of $y = f(x)$ has no "holes", no "jumps" and no vertical asymptotes at $x = a$. When answering free response questions on the AP exam, the formal definition of continuity is required. To earn all of the points on the free response question scoring rubric, all three of the following criteria need to be met, with work shown:

A function is *continuous* at a point $x = a$ if and only if:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$ (i.e., the limit equals the function value)

- *To show that a function is **continuous** show that the limit (or perhaps two one-sided limits) equals the value at the point. (See 2007 AB 6)*

Increasing/Decreasing Intervals of a Function

Remember: $f'(x)$ determines whether a function is increasing or decreasing, so always use the sign of $f'(x)$ when determining and justifying whether a function $f(x)$ is increasing or decreasing on (a, b).

Situation	Explanation
$f(x)$ is increasing on the interval (a, b)	$f(x)$ is increasing on the interval (a, b) because $f'(x) > 0$
$f(x)$ is decreasing on the interval (a, b)	$f(x)$ is decreasing on the interval (a, b) because $f'(x) < 0$

Relative Minimums/Maximums and Points of Inflection

Sign charts are very commonly used in calculus classes and are a valuable tool for students to use when testing for relative extrema and points of inflection. However, a sign chart will never earn students any points on the AP exam. Students should use sign charts when appropriate to help make determinations, but they cannot be used as a justification or explanation on the exam.

Situation (at a point $x = a$ on the function $f(x)$)	Proper Explanation/Reasoning
Relative Minimum	$f(x)$ has a relative minimum at the point $x = a$ because $f'(x)$ changes signs from negative to positive when $x = a$.
Relative Maximum	$f(x)$ has a relative maximum at the point $x = a$ because $f'(x)$ changes signs from positive to negative when $x = a$.
Point of Inflection	$f(x)$ has a point of inflection at the point $x = a$ because $f''(x)$ changes sign when $x = a$

- ***Local extreme values** may be justified by the First Derivative Test, the Second Derivative Test, or the Candidates' Test. In each case the hypotheses must be shown to be true either in the given or by the student's work.*
- ***Absolute Extreme Values** may be justified by the same three tests (often the Candidates' Test is the easiest), but here the student must consider the entire domain. This may be done (for a continuous function) by saying specifically that this is the only place where the derivative changes sign in the proper direction.*

Tangent Line Approximations

Unlike a Riemann Sum, determining whether a tangent line is an over/under approximation is not related to whether a function is increasing or decreasing. When determining (or justifying) whether a tangent line is an over or under approximation, the concavity of the function must be discussed. It is important to look at the concavity on the interval from the point of tangency to the x-value of the approximation, not just the concavity at the point of tangency.

Example Justification: The approximation of $f(1.1)$ using the tangent line of f(x) at the point $x=1$ is an over-approximation of the function because $f''(x) < 0$ on the interval $1 < x < 1.1$.

- ***Overestimates*** *or* ***underestimates*** *usually depend on the concavity between the two points used in the estimates.*

Speed Increasing/Decreasing (Particle Motion)

Many students struggle with the concept of speed in particle motion. The speed of a particle is the absolute value of velocity. If a particle's velocity and acceleration are in the same direction, then we know its speed will be increasing. In other words, if the velocity and acceleration have the same sign, then its speed is increasing. On the other hand, if the velocity and acceleration are in opposite directions (different signs), then the speed is decreasing.

When justifying an answer about whether the speed of a particle is increasing/decreasing at a given time, determine both the velocity and acceleration at that time and make reference to the signs of their values.

Answer	Possible Justification
Speed is increasing when $t=c$	Speed is increasing because $v(c)>0$ and $a(c)>0$
Speed is increasing when $t=c$	Speed is increasing because $v(c)<0$ and $a(c)<0$
Speed is decreasing when $t=c$	Speed is decreasing because $v(c)>0$ and $a(c)<0$
Speed is decreasing when $t=c$	Speed is decreasing because $v(c)<0$ and $a(c)>0$

- ***Speed*** *is increasing on intervals where the velocity and acceleration have the same sign; decreasing where they have different signs. (2013 AB 2 d)*

- *To use the* ***Mean Value Theorem*** *state that the function is continuous and differentiable on the interval and show the computation of the slope between the endpoints of the interval. (2007 AB 3 b, 2103 AB3/BC3)*

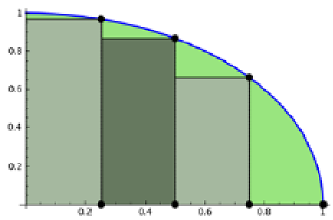
- *To use the* ***Intermediate Value Theorem*** *state that the function is continuous and show that the values at the endpoints bracket the value in question (2007 AB 3 a)*

- *For* ***L'Hôpital's Rule*** *state, in separate statements, that the limit of the numerator and denominator are either both zero or both infinite and then make the conclusion you can use l'Hopitals rule. (2013 BC 5 a)*

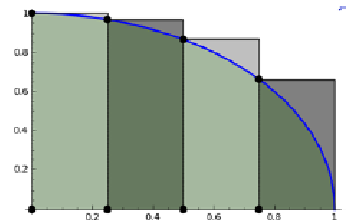
- *The* ***meaning of a derivative*** *should include the value and (1) what it is (the rate of change of …, velocity of …, slope of …), (2) the time it obtains this value, and (3) the units. (2012 AB1/BC1)*

Accumulation

Left and right Riemann sums



Right Riemann Sum



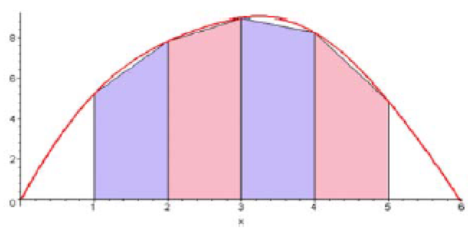
Left Riemann Sum

Correct justification for over and under approximations:

f(x)	**Left Riemann Sum**	**Right Riemann Sum**
Increasing ($f'(x) > 0$)	Under approximates the area because *f*(x) is increasing	Over approximates the area because *f*(x) is increasing
Decreasing ($f'(x) < 0$)	Over approximates the area because *f*(x) is decreasing	Under approximates the area because *f*(x) is decreasing

Incorrect Reasoning: *The left Riemann Sum is an under approximation because the rectangles are all underneath or below the graph. Stating that the rectangles are below the function is not acceptable mathematical reasoning. It merely restates that it is an under approximation but does not explain WHY.*

Trapezoidal approximations



Over/Under Approximations with Trapezoidal Approximations

f(x)	**Trapezoidal Sum**
Concave Up ($f''(x) > 0$)	Over approximates the area because $f''(x) > 0$
Concave Down ($f''(x) < 0$)	Under approximates the area because $f''(x) < 0$

Interpretation of a Definite Integral

When interpreting the meaning of a definite integral, remember the following:

1. Recognize that a definite integral gives an accumulation or total
2. Always give meaning to the integral in CONTEXT to the problem
3. Give the units of measurement
4. Reference the limits of integration with appropriate units in the context of the problem

- *The* ***meaning of a definite integral*** *should include the value and (1) what the integral gives (amount, average value, change of position), (2) the units, and (3) what the limits of integration mean. One way of determining this is to remember the Fundamental Theorem of Calculus* $\int_a^b f'(x)\,dx = f(b) - f(a)$. *The integral is the difference between whatever f represents at b and what it represents at a. (2009 AB 2 c, AB 3c, 2013 AB3/BC3 c)*