# Matching Definite Integrals to Limits of Riemann Sums

You have two sets of cards in front of you. Working with your partner, match each definite integral card to its associated limit of a Riemann sum card. Discuss the clues you are using to make the matches and how the parts in each notation relate to one another.

After you have completed the activity, answer the question below.

**Check your understanding**

Write down three things you and your partner discussed in terms of how you matched the cards with a definite integral to the cards with the limit of a Riemann sum.

# Apply Your Understanding of Definite Integrals

In the chart below, a definite integral has been provided for you. If a definite integral has been provided, write the corresponding limit of a Riemann sum.

| | Definite Integral | Limit of Riemann Sum |
|---|---|---|
| **1.** | $\int_{0}^{6} \sqrt{2x+1}\,dx$ | |
| **2.** | $\int_{-2}^{3} x^2 - 3\,dx$ | |
| **3.** | $\int_{1}^{6} 3x - 4\,dx$ | |
| **4.** | $\int_{-2}^{4} x^3\,dx$ | |
| **5.** | $\int_{-2}^{0} \sqrt{x^2+1}\,dx$ | |
| **6.** | $\int_{2}^{6} 5x + 7\,dx$ | |
| **7.** | $\int_{0}^{4} 6x^2 - 2\,dx$ | |
| **8.** | $\int_{1}^{3} 4x^3 - 1\,dx$ | |

**Check your understanding**

**1.** The question below is followed by four student responses – three are INCORRECT and only one is CORRECT. Select the correct answer choice, and identify the error in each of the incorrect responses

Question: Which of the following limits is equal to $\int_3^5 x^4\,dx$ ?

Student Response A: $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{1}{n}$

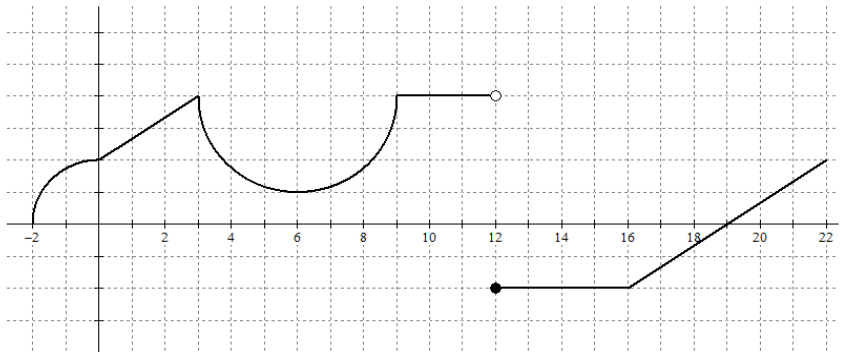
Student Response B: $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{2}{n}$

Student Response C: $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{1}{n}$

Student Response D: $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{2}{n}$

**2.** Look back at choice A. What is the definite integral associated with that limit expression?

# Translating Notation and Finding Definite Integral Values




The graph above consists of a quarter circle, a half circle and four line segments. For each of the expressions below, fill in the missing limits of Riemann sums. Then determine the value of each definite integral using geometric formulas (without using a calculator).

| Limit of Riemann Sum | Definite Integral | Value of Definite Integral |
|---|---|---|
| | $\int_{12}^{16} (-2)\,dx$ | |
| | $\int_{3}^{9} 4-\sqrt{9-(x-6)^2}\,dx$ | |
| | $\int_{0}^{3} \left(\frac{2}{3}x+2\right)dx$ | |
| | $\int_{-2}^{0} \sqrt{4-x^2}\,dx$ | |
| | $\int_{9}^{12} 4\,dx$ | |
| | $\int_{16}^{22} \frac{2}{3}(x-19)\,dx$ | |

# Additional Learning Resources: Remembering the $\Delta x$ term

A common mistake when matching definite integrals and Riemann sums is to forget or ignore the *dx* in the integral, which corresponds with the $\Delta x$ term in the Riemann sum. A common mistake looks something like this:

Given the integral $\int_a^b f(x)dx$, the corresponding Riemann sum is $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)$.

What is missing is the $\Delta x$ term. The correct Riemann sum is $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k) * \frac{b-a}{n}$, where $\Delta x = \frac{b-a}{n}$.

Try the following exercise. Each of the below definite integrals is incorrectly written as a Riemann sum; the $\Delta x$ portion is missing. Write in the correct $\Delta x$ to make the statement correct.

**1.** The Riemann sum for $\int_0^3 f(x)dx$ is $\lim_{n\to\infty}\sum_{k=1}^{n} f(\frac{3k}{n})$

**2.** The Riemann sum for $\int_0^5 g(x)dx$ is $\lim_{n\to\infty}\sum_{k=1}^{n} g(\frac{5k}{n})$

**3.** The Riemann sum for $\int_1^4 q(x)dx$ is $\lim_{n\to\infty}\sum_{k=1}^{n} q(1+\frac{3k}{n})$

# Additional Learning Resources: Determining $\Delta x$

One of the key relationships between a Riemann sum and a definite integral is the $\Delta x$ term in the Riemann sum. For $\int_a^b f(x)dx$, with Riemann sum $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$ and partitions of equal length, we calculate $\Delta x$ as $\Delta x = \frac{b-a}{n}$, where $a$ and $b$ come from the definite integral, and $n$ is the number of partitions. Note that, if the partitions are of equal length, then we can also write $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$ as $\lim_{n\to\infty}\Delta x\sum_{k=1}^{n} f(x_k)$.

Thus, if we wanted to find $\int_0^2 e^x dx$, our Riemann sum would look something like $\lim_{n\to\infty}\sum_{k=1}^{n} e^{\frac{2k}{n}} * \frac{2}{n}$, while $\int_2^7 e^x dx$ would look something like $\lim_{n\to\infty}\sum_{k=1}^{n} e^{2+\frac{5k}{n}} * \frac{5}{n}$. In the first integral, $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$, while in the second, $\Delta x = \frac{b-a}{n} = \frac{7-2}{n} = \frac{5}{n}$.You can see that there are other things going on inside the Riemann sum, but identifying the $\Delta x$ is one important part of the process.

Note that you can also write $\lim_{n\to\infty}\sum_{k=1}^{n} e^{2+\frac{5k}{n}} * \frac{5}{n}$ as $\frac{5}{n}\lim_{n\to\infty}\sum_{k=1}^{n} e^{2+\frac{5k}{n}}$.

For the following exercise, use what you know about $\Delta x$ to match the definite integral in the left column with the corresponding Riemann sum on the right.

| Definite Integral | Riemann Sum |
|---|---|
| A. $\int_0^3 f(x)dx$ | 1. $\lim_{n\to\infty}\sum_{k=1}^{n} f(2+\frac{4k}{n}) * \frac{4}{n}$ |
| B. $\int_1^2 f(x)dx$ | 2. $\lim_{n\to\infty}\sum_{k=1}^{n} f(\frac{3k}{n}) * \frac{3}{n}$ |
| C. $\int_2^6 f(x)dx$ | 3. $\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n} f(-1+\frac{2k}{n})$ |
| D. $\int_{-1}^1 f(x)dx$ | 4. $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(1+\frac{k}{n})$ |

# Additional Learning Resources: Representing $x_k$ as $x_0 + k\Delta x$

Recall that the Riemann sum for the definite integral $\int_a^b f(x)dx$ is $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$. That $f(x_k)$ piece is also sometimes written as $f(a + k\Delta x)$. We know that $\Delta x = \frac{b-a}{n}$, and *k* is the letter we're using in the Riemann sum for the partitions (on the bottom and top of the sigma, you can see that *k* starts at 1 and iterates all the way to *n*). The *a* is where the integral starts, i.e. the lower limit of integration.

So imagine that we want to write $\int_1^3 f(x)dx$ as the limit of a Riemann sum. We have the model of $\lim_{n\to\infty}\sum_{k=1}^{n} f(a + k\Delta x)\Delta x$ to use. We know that *a* = 1, and we can calculate $\Delta x = \frac{2}{n}$, so our expression will be $\lim_{n\to\infty}\sum_{k=1}^{n} f(1 + k\frac{2}{n})\frac{2}{n}$, or more simply, $\lim_{n\to\infty}\sum_{k=1}^{n} f(1 + \frac{2k}{n})\frac{2}{n}$.

Now imagine that we want to write $\int_1^3 f(x)dx$ as the limit of a Riemann sum, and we know that $f(x) = x^3$. We start exactly the same as before: we have the model of $\lim_{n\to\infty}\sum_{k=1}^{n} f(a + k\Delta x)\Delta x$ to use. We know that *a* = 1, and we can calculate $\Delta x = \frac{2}{n}$, so our expression will be $\lim_{n\to\infty}\sum_{k=1}^{n} f(1 + k\frac{2}{n})\frac{2}{n}$, or more simply, $\lim_{n\to\infty}\sum_{k=1}^{n} f(1 + \frac{2k}{n})\frac{2}{n}$. And since $f(x) = x^3$, our Riemann sum becomes $\lim_{n\to\infty}\sum_{k=1}^{n} (1 + \frac{2k}{n})^3\frac{2}{n}$.

Try the following exercises. For the given definite integral, write the appropriate limit of a Riemann sum by filling in the correct information in the box

**1.** $\int_0^4 f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\square)\frac{4}{n}$

**2.** $\int_2^9 f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\square)\frac{7}{n}$

**3.** $\int_{-2}^3 f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\square)\frac{5}{n}$

Try the following exercises. For the given definite integral and the given function, write the appropriate limit of a Riemann sum by replacing the box with correct notation.

**4.** $\int_0^4 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \square \frac{4}{n}$; $f(x) = x^4$

**5.** $\int_2^9 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \square \frac{7}{n}$; $f(x) = 3x^2$

**6.** $\int_{-2}^3 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \square \frac{5}{n}$; $f(x) = \sin(4x^2)$