EXERCISE 1

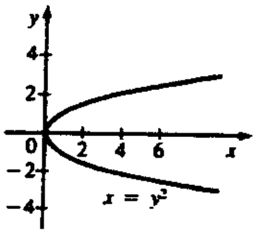
1. (a) $a = r\theta \Rightarrow a = 10\times 2.5 = 25\,\text{cm}$ (b) $\theta = \frac{a}{r} = \frac{12}{10} = 1.2$

   (c) $A = \frac{1}{2}ar = \frac{1}{2}\times 20\times 10 = 100\,\text{cm}^2$ (d) $A = \frac{1}{2}r^2\theta = \frac{1}{2}\times 100\times\frac{2}{3}\pi = \frac{100\pi}{3}\,\text{cm}^2$

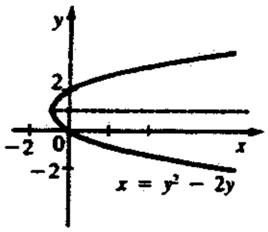
2. $A = \frac{1}{2}ar \Rightarrow a = \frac{2A}{r} = \frac{2\pi}{6} = \frac{\pi}{3}\,\text{cm}$ $\theta = \frac{a}{r} = \frac{\frac{\pi}{3}}{6} = \frac{\pi}{18}$

EXERCISE 2.

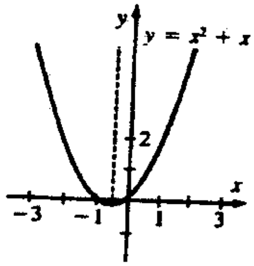
1. (a) $x = y^2$ is not a function



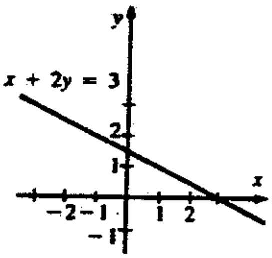
(b) $x = y^2 - 2y$ is not a function



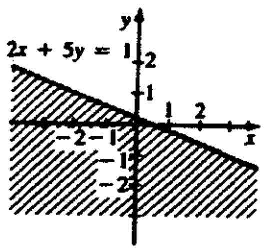
(c) $y = x^2 + x$ is a function



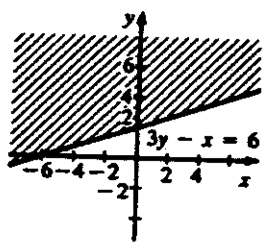
(d) $x + 2y = 3$ is a function



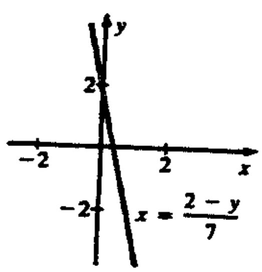
(e) $2x + 5y < 1$ is not a function



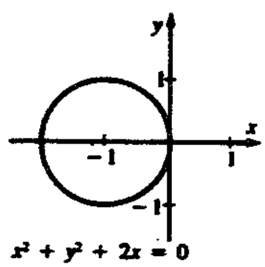
(f) $3y - x \geq 6$ is not a function



(g) $x = \frac{2-y}{7}$ is a function



(h) $x^2 + y^2 + 2x = 0$ is not a function



2. (a) $f(x)=x+1$ is $1-1$

(b) $g(x)=|x|$ is not $1-1$

(c) $y=3-2x$ is $1-1$

(d) $h(x)=\frac{1}{x}$ is $1-1$

(e) $F(x)=\frac{1}{x^2}$ is not $1-1$

(f) $y=1-x^2$ is not $1-1$

(g) $f(t)=-t^3$ is $1-1$

(h) $f(t)=t^4$ is not $1-1$

(i) $y=\sqrt{x}$ is $1-1$

(j) $f(x)=\frac{1}{x^2}$, $x<0$ is $1-1$

3. (a) $y=\frac{1}{2}(x-7)$ has inverse $2x=y-7 \Rightarrow y=2x+7$

(b) $y=\frac{1}{5}(36-x)$ has inverse $5x=36-y \Rightarrow y=36-5x$

(c) $y=5x^3-6$ has inverse $x=5y^3-6 \Rightarrow 5y^3=x+6 \Rightarrow y=\sqrt[3]{\frac{x+6}{5}}$

(d) $y=\sqrt{x}$ has inverse $x=\sqrt{y} \Rightarrow y=x^2,\ x\geq 0$

(e) $y=\sqrt{x-3}$ has inverse $x=\sqrt{y-3} \Rightarrow x^2=y-3 \Rightarrow y=x^2+3,\ x\geq 0$

(f) $y=1+\frac{1}{x}$ has inverse $x=1+\frac{1}{y} \Rightarrow \frac{1}{y}=x-1 \Rightarrow y=\frac{1}{x-1}$

(g) $y=\frac{1}{1+x}$ has inverse $x=\frac{1}{1+y} \Rightarrow 1+y=\frac{1}{x} \Rightarrow y=\frac{1}{x}-1$

(h) $y=\frac{1-x}{1+x}$ has inverse $x=\frac{1-y}{1+y} \Rightarrow x+xy=1-y \Rightarrow xy+y=1-x$

$\Rightarrow y=\frac{1-x}{1+x}$

(i) $y=\frac{4x-1}{3x+2}$ has inverse $x=\frac{4y-1}{3y+2} \Rightarrow 3xy+2x=4y-1$

$\Rightarrow y(3x-4)=-2x-1 \Rightarrow y=\frac{2x+1}{4-3x}$

(j) $y=\frac{\pi-3x}{x}$ has inverse $x=\frac{\pi-3y}{y} \Rightarrow xy=\pi-3y \Rightarrow y(x+3)=\pi$

$\Rightarrow y=\frac{\pi}{x+3}$

(k) $y = x^4, x \geq 0$ has inverse $x = y^4,\ y \geq 0 \Rightarrow y = x^{\frac{1}{4}}$.

(l) $y = 3(x-1)^2,\ x \geq 1$ has inverse $x = 3(y-1)^2,\ y \geq 1 \Rightarrow (y-1)^2 = \frac{x}{3},\ y \geq 1$

$\Rightarrow y = \sqrt{\frac{x}{3}} + 1$

(m) $y = \sqrt{x^2+9},\ x \geq 0$ has inverse $x = \sqrt{y^2+9},\ y \geq 0,$

$\Rightarrow x^2 = y^2 + 9,\ y \geq 0 \Rightarrow y = \sqrt{x^2-9}$

(n) $y = \sqrt{25-x^2},\ x \leq 0$ has inverse $x = \sqrt{25-y^2},\ y \leq 0$

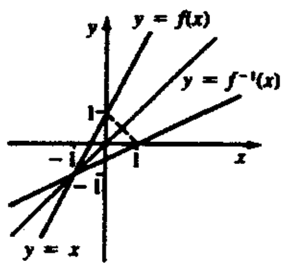
$\Rightarrow x^2 = 25 - y^2,\ y \leq 0,\ x \geq 0 \Rightarrow y^2 = 25 - x^2,\ y \leq 0,\ x \geq 0$

$\Rightarrow y = -\sqrt{25-x^2},\ x \geq 0$

4. (a) $f(x) = 2x+1$

$\Rightarrow y = 2x+1$ and

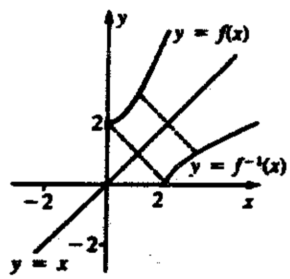
has inverse $x = 2y+1 \Rightarrow y = \frac{x-1}{2}$



(b) $f(x) = x^2+2, x \geq 0$

$\Rightarrow y = x^2+2,\ x \geq 0$

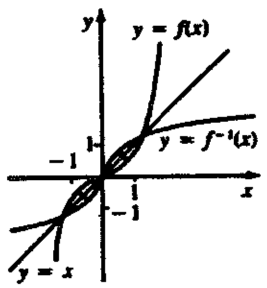
has inverse $x = y^2+2,\ y \geq 0 \Rightarrow y = \sqrt{x-2}$



(c) $f(x) = x^3$

$\Rightarrow y = x^3$ and

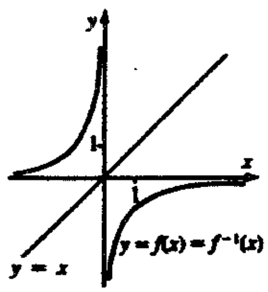
has inverse $x = y^3 \Rightarrow y = x^{\frac{1}{3}}$



(d) $f(x) = -\frac{1}{x}$

$\Rightarrow y = -\frac{1}{x}$ and

has inverse $xy = -1 \Rightarrow y = -\frac{1}{x}$



EXERCISE 7.1

1. If $x = 0.0001$, $\frac{\sin 3x}{x} = 2.999999955$ 2. If $x = 0.0001$, $\frac{\sin 2x}{\sin 3x} = 0.666666672$

3. If $x = 0.0001$, $\frac{\sin^3 2x}{\sin^3 3x} = 0.296296304$ 4. If $x = 0.0001$, $\frac{1-\cos^2 x}{x^2} = 0.998$

5. If $x = 0.0001$, $\frac{1-\cos x}{\tan x} = 0.0000499$ 6. If $x = 0.0001$, $\frac{\sin(\cos x)}{\sec x} = 0.841470978$

7. $\lim_{x\to 0} \frac{\sin 3x}{x} = 3 \lim_{3x\to 0} \frac{\sin 3x}{3x} = 3\times 1 = 3$

8. $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \dfrac{ax \lim_{ax\to 0} \frac{\sin ax}{ax}}{bx \lim_{bx\to 0} \frac{\sin bx}{bx}} = \dfrac{a \lim_{ax\to 0} \frac{\sin ax}{ax}}{b \lim_{bx\to 0} \frac{\sin bx}{bx}} = \frac{a\times 1}{b\times 1} = \frac{a}{b}$

9. $\lim_{x\to 0} \frac{\sin^3 2x}{\sin^3 3x} = \left(\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}\right)^3 = \left(\dfrac{2x \lim_{2x\to 0} \frac{\sin 2x}{2x}}{3x \lim_{3x\to 0} \frac{\sin 3x}{3x}}\right)^3 = \left(\frac{2\times 1}{3\times 1}\right)^3 = \frac{8}{27}$

10. $\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0} \sin x \lim_{x\to 0} \frac{\sin x}{x}$

$\lim_{x\to 0} \frac{1}{1+\cos x} = 0\times 1\times \frac{1}{2} = 0$

11. $\lim_{x\to 0} (x^2+\cos x) = \lim_{x\to 0} x^2 + \lim_{x\to 0} \cos x = 0+\cos 0 = 0+1 = 1$

12. $\lim_{x\to \frac{\pi}{3}} (\sin x - \cos x) = \lim_{x\to \frac{\pi}{3}} \sin x - \lim_{x\to \frac{\pi}{3}} \cos x = \sin\frac{\pi}{3} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$

13. $\lim_{x\to \frac{\pi}{4}} \frac{\sin x}{3x} = \dfrac{\sin\frac{\pi}{4}}{\frac{3\pi}{4}} = \frac{1}{\sqrt{2}}\times\frac{4}{3\pi} = \frac{2\sqrt{2}}{3\pi}$

14. $\lim_{x\to -3\pi} x^3 \sin^4 x = (-3\pi)^3(\sin^4(-3\pi)) = -27\pi^3\times 0 = 0$

15. $\lim_{x\to 0} \frac{\sin 5x}{5} = \frac{\sin 0}{5} = \frac{0}{5} = 0$

16. $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\tan x}{4x} = \dfrac{\tan\frac{\pi}{4}}{\pi} = \dfrac{1}{\pi}$

17. $\lim\limits_{x\to 0} \dfrac{\tan 3x}{3\tan 2x} = \lim\limits_{x\to 0} \dfrac{\frac{\sin 3x}{\cos 3x}}{\frac{3\sin 2x}{\cos 2x}} = \dfrac{1}{3} \lim\limits_{x\to 0} \dfrac{\cos 2x}{\cos 3x} \dfrac{3x \lim\limits_{3x\to 0} \frac{\sin 3x}{3x}}{2x \lim\limits_{2x\to 0} \frac{\sin 2x}{2x}}$

$= \dfrac{1}{3} \times 1 \times \dfrac{3 \lim\limits_{3x\to 0} \frac{\sin 3x}{3x}}{2 \lim\limits_{2x\to 0} \frac{\sin 2x}{2x}} = \dfrac{1}{3} \times \dfrac{3(1)}{2(1)} = \dfrac{1}{2}$

18. $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{x^2} = 9\Big(\lim\limits_{3x\to 0} \dfrac{\sin 3x}{3x}\Big)^2 = 9 \times 1^2 = 9$

19. $\lim\limits_{x\to\frac{\pi}{6}} \sqrt{\sin x} = \sqrt{\sin\frac{\pi}{6}} = \sqrt{\frac{1}{2}}$

20. $\lim\limits_{x\to 0} \dfrac{\sin 6x}{\cos 4x} = \dfrac{\sin 0}{\cos 0} = \dfrac{0}{1} = 0$

21. $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\sin x} = \lim\limits_{x\to 0} \dfrac{\cos^2 x - 1}{\sin x(\cos x + 1)}$

$= \lim\limits_{x\to 0} \dfrac{-\sin^2 x}{\sin x(\cos x + 1)} = \lim\limits_{x\to 0} \dfrac{-\sin x}{\cos x + 1} = -\dfrac{0}{1+1} = 0$

22. $\lim\limits_{x\to 0} \dfrac{\tan x}{4x} = \lim\limits_{x\to 0} \dfrac{\sin x}{4x\cos x} = \dfrac{1}{4} \lim\limits_{x\to 0} \dfrac{\sin x}{x} \lim\limits_{x\to 0} \dfrac{1}{\cos x} = \dfrac{1}{4} \times 1 \times \dfrac{1}{1} = \dfrac{1}{4}$

23. $\lim\limits_{x\to 0} \dfrac{x^3}{\tan^3 2x} = \Big(\lim\limits_{x\to 0} \dfrac{x\cos 2x}{\sin 2x}\Big)^3 = \Big(\dfrac{1}{2} \lim\limits_{2x\to 0} \dfrac{2x}{\sin 2x} \lim\limits_{x\to 0} \cos 2x\Big)^3 = \Big(\dfrac{1}{2} \times 1 \times 1\Big)^3 = \dfrac{1}{8}$

24. $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{2x^2} = \lim\limits_{x\to 0} \dfrac{1 - \cos^2 x}{2x^2(1 + \cos x)} = \dfrac{1}{2} \lim\limits_{x\to 0} \dfrac{\sin^2 x}{x^2(1 + \cos x)}$

$= \dfrac{1}{2}\Big(\lim\limits_{x\to 0} \dfrac{\sin x}{x}\Big)^2 \lim\limits_{x\to 0} \dfrac{1}{1 + \cos x} = \dfrac{1}{2} \times 1^2 \times \dfrac{1}{1+1} = \dfrac{1}{4}$

25. $\lim_{x \to 0} \frac{x}{\sin\frac{x}{2}} = 2 \lim_{\frac{x}{2} \to 0} \frac{\frac{x}{2}}{\sin\frac{x}{2}} = 2 \times 1 = 2$

26. $\lim_{x \to 0} \frac{2\tan x}{x\sec x} = 2 \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x\frac{1}{\cos x}} = 2 \lim_{x \to 0} \frac{\sin x}{x} = 2 \times 1 = 2$

27. $\lim_{x \to 0} \frac{1-\cos^2 x}{x^2} = \lim_{x \to 0} \frac{\sin^2 x}{x^2} = \left(\lim_{x \to 0} \frac{\sin x}{x} \right)^2 = 1^2 = 1$

28. $\lim_{x \to 0} \frac{1-\cos 2x}{x^2} = \lim_{x \to 0} \frac{1-\cos^2 2x}{x^2(1+\cos 2x)} = \left(2 \lim_{x \to 0} \frac{\sin 2x}{2x}\right)^2 \lim_{x \to 0} \frac{1}{1+\cos 2x}$

$= 2^2 \times \frac{1}{1+1} = 2$

29. $\lim_{x \to \frac{\pi}{2}} \frac{\cot x}{\frac{\pi}{2} - x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{(\frac{\pi}{2} - x)\sin x} = \lim_{x - \frac{\pi}{2} \to 0} \frac{-\sin(x - \frac{\pi}{2})}{-(x - \frac{\pi}{2})\sin x}$

$= \lim_{x - \frac{\pi}{2} \to 0} \frac{\sin(x - \frac{\pi}{2})}{(x - \frac{\pi}{2})} \times \lim_{x \to \frac{\pi}{2}} \frac{1}{\sin x} = 1 \times 1 = 1$

30. $\lim_{x \to \pi} \frac{\sin x}{x - \pi} = -\lim_{x - \pi \to 0} \frac{\sin(x - \pi)}{x - \pi} = -1$

31. $\lim_{x \to 0} \frac{\sin^2 x \cos x}{1 - \cos x} = \lim_{x \to 0} \frac{\sin^2 x \cos x(1 + \cos x)}{1 - \cos^2 x} = \lim_{x \to 0} \cos x(1 + \cos x) = 1(1+1) = 2$

32. $\lim_{x \to 0} \frac{\sin x}{\tan x} = \lim_{x \to 0} \cos x = 1$

33. $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x\cos x} = \lim_{x \to 0} \frac{2\sin x - 2\sin x\cos x}{x\cos x} = \lim_{x \to 0} \frac{2\sin x(1 - \cos x)}{x\cos x}$

$= 2 \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \frac{1 - \cos x}{\cos x} = 2 \times 1 \times \frac{1-1}{1} = 0$

34. $\lim_{x\to 0} \frac{\tan x - \sin x}{x\cos x} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x\cos x} = \lim_{x\to 0} \frac{\sin x - \sin x\cos x}{x\cos^2 x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1-\cos x}{\cos^2 x} = 1 \times \frac{1-1}{1^2} = 0$

35. $\lim_{x\to 0} \frac{1-\cos x}{\tan x} = \lim_{x\to 0} \frac{1-\cos^2 x}{\frac{\sin x}{\cos x}(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x\cos x}{\sin x(1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin x\cos x}{1+\cos x} = \frac{0\times 1}{1+1} = 0$

36. $\lim_{x\to 0} \frac{\csc x - \cot x}{\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin^2 x} = \lim_{x\to 0} \frac{(1-\cos^2 x)}{\sin^2 x(1+\cos x)} = \lim_{x\to 0} \frac{1}{1+\cos x} = \frac{1}{1+1} = \frac{1}{2}$

37. $\lim_{x\to 0} \frac{\sin 2x}{2x^2+x} = \lim_{x\to 0} \frac{2\sin x\cos x}{x(2x+1)} = 2\lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1}{2x+1} = 2\times 1\times \frac{1}{0+1} = 2$

38. $\lim_{x\to 0} \frac{\sin(\cos x)}{\sec x} = \frac{\sin 1}{\sec 0} = \frac{\sin 1}{1} = \sin 1$

39. (a) If $x = 0.1$, $\frac{\tan x - x}{x^3} = 0.334\ 672\ 085$. If $x = 0.01$, $\frac{\tan x - x}{x^3} = 0.3333466$.

If $x = 0.001$, $\frac{\tan x - x}{x^3} = 0.33333$. If $x = 0.0001$, $\frac{\tan x - x}{x^3} = 0.333$

(b) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$ appears to be approaching 0.3

(c) If $x = 0.00001$, $\frac{\tan x - x}{x^3} = 0.3$. If $x = 0.000001$, $\frac{\tan x - x}{x^3} = -10$

If $x = 0.0000001$, $\frac{\tan x - x}{x^3} = -1000$. The correct value of this limit is $\frac{1}{3}$.

If we had let $x = 1, 0.5, 0.1, 0.05, 0.01, 0.005$, our initial guess would have been $\frac{1}{3}$. Eventually all calculators will give incorrect values. Different calculators will give different incorrect values. Because of loss of significant digits in the process of rounding off , when two numbers that are very close together are subtracted, this type of error is likely to occur.

40. If $x > 0$, $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$. If $x < 0$, $\lim_{x \to 0^-} \frac{\sin x}{-x} = -1$. Since the left and right hand limits have different values the $\lim_{x \to 0} \frac{\sin x}{|x|}$ does not exist.

41. $\lim_{x \to 0} \frac{\sin x}{x + \sin x} = \lim_{x \to 0} \frac{1}{\frac{x + \sin x}{\sin x}} = \lim_{x \to 0} \frac{1}{\frac{x}{\sin x} + 1} = \frac{1}{1+1} = \frac{1}{2}$

42. $\lim_{x \to 1^-} \frac{\sin(x-1)}{|x-1|} = \lim_{x \to 1^-} \frac{\sin(x-1)}{1-x} = -\lim_{x-1 \to 0^-} \frac{\sin(x-1)}{x-1} = -1$

43. $\lim_{h \to 0} \frac{\sin(a+h) - \sin a}{h} = \lim_{h \to 0} \frac{\sin a \cos h + \cos a \sin h - \sin a}{h}$

$= \lim_{h \to 0} \frac{\sin a(\cos h - 1) + \cos a \sin h}{h} = \sin a \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos a \lim_{h \to 0} \frac{\sin h}{h}$

$= \sin a \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} + \cos a(1) = \sin a \lim_{h \to 0} \frac{-\sin^2 h}{h(\cos h + 1)} + \cos a$

$= -\sin a \lim_{h \to 0} \frac{\sin h}{h} \lim_{h \to 0} \frac{\sin h}{\cos h + 1} + \cos a = -\sin a \times 1 \times 0 + \cos a = \cos a$

44. $\lim_{h \to 0} \frac{\cos(a+h) - \cos a}{h} = \lim_{h \to 0} \frac{\cos a \cos h - \sin a \sin h - \cos a}{h}$

$= \lim_{h \to 0} \frac{\cos a(\cos h - 1) - \sin a \sin h}{h} = \cos a \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin a \lim_{h \to 0} \frac{\sin h}{h}$

$= \cos a(0) - \sin a(1) = -\sin a$

EXERCISE 7.2

1. (a) $\frac{dy}{dx} = -\sin(-4x)(-4) = 4\sin(-4x) = -4\sin(4x)$

(b) $\frac{dy}{dx} = \cos(3x + 2\pi)(3) = 3\cos(3x + 2\pi)$

(c) $\frac{dy}{dx} = 4\cos(-2x^2 - 3)(-4x) = -16x\cos(-2x^2 - 3)$

(d) $\frac{dy}{dx} = \frac{1}{2}\sin(4 + 2x)(2) = \sin(4 + 2x)$

(e) $\frac{dy}{dx} = \cos x^2(2x) = 2x\cos x^2$

(f) $\frac{dy}{dx} = -(-\sin x^2)(2x) = 2x\sin x^2$

(g) $\frac{dy}{dx} = -2\sin^{-3}(x^3)\cos x^3(3x^2) = \frac{-6x^2\cos x^3}{\sin^3 x^3}$

(h) $\frac{dy}{dx} = -\sin(x^2-2)^2[2(x^2-2)2x] = -4x(x^2-2)\sin(x^2-2)^2$

(i) $\frac{dy}{dx} = 12\sin^3(2-x)^{-1}\cos(2-x)^{-1}(-1)(2-x)^{-2}(-1)$

$= \frac{12\sin^3(2-x)^{-1}\cos(2-x)^{-1}}{(2-x)^2}$

(j) $\frac{dy}{dx} = x(-\sin x) + \cos x(1) = \cos x - x\sin x$

(k) $\frac{dy}{dx} = \frac{\sin x(1) - x(\cos x)}{\sin^2 x} = \frac{\sin x - x\cos x}{\sin^2 x}$

(l) $\frac{dy}{dx} = \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$

$= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}$

(m) $\frac{dy}{dx} = 6(1+\cos^2 x)^5(2\cos x)(-\sin x) = -12\sin x\cos x(1+\cos^2 x)^5$

$= -6\sin 2x(1+\cos^2 x)^5$

(n) $\frac{dy}{dx} = \cos\frac{1}{x}(-\frac{1}{x^2}) = -\frac{1}{x^2}\cos\frac{1}{x}$

(o) $\frac{dy}{dx} = \cos(\cos x)(-\sin x) = -\sin x \cos(\cos x)$

(p) $\frac{dy}{dx} = 3\cos^2(\sin x)[-\sin(\sin x)]\cos x = -3\cos x \sin(\sin x)\cos^2(\sin x)$

(q) $\frac{dy}{dx} = x(-\sin\frac{1}{x})(-\frac{1}{x^2}) + \cos\frac{1}{x}(1) = \frac{1}{x}\sin\frac{1}{x} + \cos\frac{1}{x}$

(r) $\frac{dy}{dx} = \frac{\cos x(2\sin x\cos x) - \sin^2 x(-\sin x)}{\cos^2 x} = \frac{\sin x(2\cos^2 x + \sin^2 x)}{\cos^2 x}$

(s) $\frac{dy}{dx} = \frac{(1-\sin 2x)\cos x - (1+\sin x)(-\cos 2x)2}{(1-\sin 2x)^2}$

$= \frac{\cos x - \cos x\sin 2x + 2\cos 2x + 2\sin x\cos 2x}{(1-\sin 2x)^2}$

(t) $\frac{dy}{dx} = 3\sin^2 x\cos x + 3\cos^2 x(-\sin x) = 3\sin x\cos x(\sin x - \cos x)$

(u) $\frac{dy}{dx} = 2\cos\left[\frac{1-\sqrt{x}}{1+\sqrt{x}}\right]\left[-\sin\left[\frac{1-\sqrt{x}}{1+\sqrt{x}}\right]\right]\frac{(1+\sqrt{x})(-\frac{1}{2\sqrt{x}}) - (1-\sqrt{x})(\frac{1}{2\sqrt{x}})}{(1+\sqrt{x})^2}$

$= \sin 2\left[\frac{1-\sqrt{x}}{1+\sqrt{x}}\right]\left(\frac{1}{\sqrt{x}(1+\sqrt{x})^2}\right)$

2. (a) $\cos y\frac{dy}{dx} = -\sin 2x(2) \Rightarrow \frac{dy}{dx} = \frac{-2\sin 2x}{\cos y}$

(b) $x(-\sin y\frac{dy}{dx}) + \cos y = +\cos(x+y)(1+\frac{dy}{dx})$

$\Rightarrow -x\sin y\frac{dy}{dx} - \cos(x+y)\frac{dy}{dx} = -\cos y + \cos(x+y)$

$\Rightarrow \frac{dy}{dx} = \frac{\cos y - \cos(x+y)}{x\sin y + \cos(x+y)}$

(c) $\cos y\frac{dy}{dx} + \frac{dy}{dx} = -\sin x + 1 \Rightarrow \frac{dy}{dx} = \frac{1-\sin x}{1+\cos y}$

(d) $\cos(\cos x)(-\sin x) = -\sin(\sin y)(\cos y\frac{dy}{dx}) \Rightarrow \frac{dy}{dx} = \frac{\sin x\cos(\cos x)}{\cos y\sin(\sin y)}$

(e) $\sin x(-\sin y\frac{dy}{dx}) + \cos y\cos x + \cos x\cos y\frac{dy}{dx} + \sin y(-\sin x) = 0$

$\Rightarrow (-\sin x\sin y + \cos x\cos y)\frac{dy}{dx} = \sin x\sin y - \cos x\cos y$

$\Rightarrow \frac{dy}{dx} = \frac{-(\cos x\cos y - \sin x\sin y)}{\cos x\cos y - \sin x\sin y} = -1$

(f) $\cos x - \sin 2x(2) = 2x\frac{dy}{dx} + 2y \Rightarrow \frac{dy}{dx} = \frac{\cos x - 2\sin 2x - 2y}{2x}$

3. (a) $\frac{dy}{dx} = 2\cos x$. The slope of the tangent is $2\cos\frac{\pi}{6} = \sqrt{3}$. The equation of the tangent is $y - 1 = \sqrt{3}(x - \frac{\pi}{6}) \Rightarrow 6y - 6 = 6\sqrt{3}x - \pi\sqrt{3}$

$\Rightarrow 6\sqrt{3}x - 6y - \pi\sqrt{3} + 6 = 0.$

(b) $\frac{dy}{dx} = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$

The slope of the tangent is $\frac{1}{\cos^2\frac{\pi}{4}} = \frac{1}{\frac{1}{2}} = 2$. The equation of the tangent is

$y - 1 = 2(x - \frac{\pi}{4}) \Rightarrow 2y - 2 = 4x - \pi \Rightarrow 4x - 2y + 2 - \pi = 0.$

(c) $\frac{dy}{dx} = \frac{\sin x}{\cos^2 x} + 2\sin x$. The slope of the tangent is $\frac{\sin\frac{\pi}{3}}{\cos^2\frac{\pi}{3}} + 2\sin\frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{4}} + \sqrt{3}$

$= 3\sqrt{3}$. The equation of the tangent is $y - 1 = 3\sqrt{3}(x - \frac{\pi}{3})$

$\Rightarrow y - 1 = 3\sqrt{3}x - \pi\sqrt{3} \Rightarrow 3\sqrt{3}x - y + 1 - \pi\sqrt{3} = 0.$

(d) $\frac{dy}{dx} = \frac{\sin^2 x(-2\cos x\sin x) - \cos^2 x(2\sin x\cos x)}{\sin^4 x}$

$= \frac{-2\cos x\sin x(\sin^2 x + \cos^2 x)}{\sin^4 x} = -\frac{2\cos x}{\sin^3 x}.$

The slope of the tangent is $-\frac{2\cos\frac{\pi}{4}}{\sin^3\frac{\pi}{4}} = -\frac{\sqrt{2}}{\frac{1}{2\sqrt{2}}} = -4.$

The equation of the tangent is $y - 1 = -4(x - \frac{\pi}{4})$
$\Rightarrow y - 1 = -4x + \pi \Rightarrow 4x + y - 1 - \pi = 0.$

(e) $\frac{dy}{dx} = \cos x - 2\sin 2x$. The slope of the tangent is $\cos\frac{\pi}{6} - 2\sin\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$. The equation of the tangent is $y - 1 = -\frac{\sqrt{3}}{2}(x - \frac{\pi}{6})$

$\Rightarrow 12y - 12 = -6\sqrt{3}x + \sqrt{3}\pi \Rightarrow 6\sqrt{3}x + 12y - 12 - \sqrt{3}\pi = 0$.

(f) $\frac{dy}{dx} = -\sin(\cos x)(-\sin x) = \sin x \sin(\cos x)$. The slope of the tangent is

$\sin\frac{\pi}{2}\sin(\cos\frac{\pi}{2}) = 1\sin 0 = 0$. When $x = \frac{\pi}{2}$, $y = \cos(\cos\frac{\pi}{2}) = \cos 0 = 1$.
The equation of the tangent is $y = 1$.

4. (a) $y' = 2\sin x\cos x = \sin 2x$. For critical numbers $\sin 2x = 0$, $-2\pi \le 2x \le 2\pi$.
$2x = -2\pi, -\pi, 0, \pi, 2\pi \Rightarrow x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$.

Interval	sin x	cos x	y'	y
$-\pi < x < -\frac{\pi}{2}$	−	−	+	increasing
$-\frac{\pi}{2} < x < 0$	−	+	−	decreasing
$0 < x < \frac{\pi}{2}$	+	+	+	increasing
$\frac{\pi}{2} < x < \pi$	+	−	−	decreasing

When $x = -\frac{\pi}{2}$, $y = \sin^2(-\frac{\pi}{2}) = 1$ is a local maximum.
When $x = 0$, $y = \sin^2 0 = 0$ is a local minimum.
When $x = \frac{\pi}{2}$, $y = \sin^2\frac{\pi}{2} = 1$ is a local maximum.

(b) $y' = -\sin x - \cos x$, $-\pi \le x \le \pi$. For critical numbers $-\sin x - \cos x = 0$
$\Rightarrow \sin x = -\cos x \Rightarrow \tan x = -1 \Rightarrow x = -\frac{\pi}{4}, \frac{3\pi}{4}$.

Interval	$-\sin x - \cos x$	y'	y
$-\pi < x < -\frac{\pi}{4}$	+	+	increases
$-\frac{\pi}{4} < x < \frac{3\pi}{4}$	−	−	decreases
$\frac{3\pi}{4} < x < \pi$	+	+	increases

When $x = -\frac{\pi}{4}$, $y = \cos(-\frac{\pi}{4}) - \sin(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$ is a local maximum.
When $x = \frac{3\pi}{4}$, $y = \cos\frac{3\pi}{4} - \sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$ is a local minimum.

5. (a) $y' = -2\sin x + 2\cos 2x$ and $y'' = -2\cos x - 4\sin 2x$. We set the second derivative equal to 0 and set up a chart to determine concavity.
$-2\cos x - 8\sin x\cos x = 0 \Rightarrow -2\cos x(1 + 4\sin x) = 0$

$\Rightarrow \cos x = 0$ and $x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\sin x = -0.25$ and $x = 3.394, 6.030$

Interval	y''	y
$0 < x < \frac{\pi}{2}$	$-$	concave down
$\frac{\pi}{2} < x < 3.394$	$+$	concave up
$3.394 < x < \frac{3\pi}{2}$	$-$	concave down
$\frac{3\pi}{2} < x < 6.030$	$+$	concave up
$6.030 < x < 2\pi$	$-$	concave down

The points of inflection are $\left(\frac{\pi}{2}, 0\right), \left(3.394, -1.453\right), \left(\frac{3\pi}{2}, 0\right)$ and $(6.030, 1.453)$

(b) $y' = 8\sin x\cos x = 4\sin 2x$ and $y'' = 8\cos 2x$. We set the second derivative equal to zero and set up a chart to determine concavity.

$8\cos 2x = 0, \ -2\pi < x < 2\pi \Rightarrow 2x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

Interval	y''	y
$-\pi < x < -\frac{3\pi}{4}$	$+$	concave up
$-\frac{3\pi}{4} < x < -\frac{\pi}{4}$	$-$	concave down
$-\frac{\pi}{4} < x < \frac{\pi}{4}$	$+$	concave up
$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$-$	concave down
$\frac{3\pi}{4} < x < \pi$	$+$	concave up

The points of inflection are $\left(-\frac{3\pi}{4}, 1\right), \left(-\frac{\pi}{4}, 1\right), \left(\frac{\pi}{4}, 1\right)$ and $\left(\frac{3\pi}{4}, 1\right)$

6. (a) A. Domain. The restricted domain $0 \le x \le 2\pi$ is given.

B. Intercepts. $f(0) = 0 + \sin 0 = 0$ is the y-intercept. The x- intercept occurs when $\sin x = -x$. Therefore the x-intercept is also 0.

C. Symmetry. Since $f(-x) = -x + \sin(-x) = -x - \sin x = -f(x)$, the function is odd and symmetric about the origin. This does not apply within our domain.

D. Asymptotes. None.

E. Intervals of Increase or Decrease. $y' = 1 - \cos x$. Since $-1 \le \cos x \le 1$, $1 - \cos x \ge 0$. Therefore the curve is always increasing.

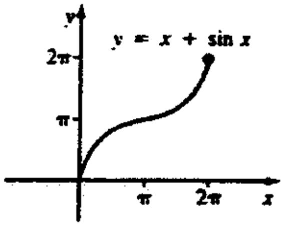
F. Local Maximum and Minimum values. Since the curve is always increasing we examine the end points of the domain. When $x = 0$, the minimum value of $y = 0$. When $x = 2\pi$, the maximum value of $y = 2\pi$.

G. Concavity and Points of Inflection. $y'' = -\sin x$. If $\sin x = 0$, $x = 0, \pi, 2\pi$.

Interval	$f''(x)$	$f(x)$
$0<x<\pi$	$-$	concave down
$\pi<x<2\pi$	$+$	concave up

When $x=\pi$ the point of inflection is (π,π)

H. Draw the graph



(b) A. Domain. The restricted domain $0<x<\pi$ is given.

B. Intercepts. $f(0)=0$ is the y-intercept. If $x\cos x=0$, $x=0$ or $\cos x=0$ and the x-intercepts are 0 and $\frac{\pi}{2}$

C. Symmetry. Since $f(-x)=-x\cos(-x)=-x\cos x=-f(x)$ the function is odd and symmetric about the origin but this does not apply within our domain.

D. Asymptotes. None.

E. Intervals of Increase and Decrease. $y'=-x\sin x+\cos x$. The critical numbers occur when $\cos x=x\sin x \Rightarrow \tan x=\frac{1}{x} \Rightarrow x\doteq 0.86$. (Trial and error using the graphs of $y=\frac{1}{x}$ and $y=\tan x$ and a calculator.)

Interval	y'	y
$0<x<0.86$	$+$	increasing
$0.86<x<\pi$	$-$	decreasing

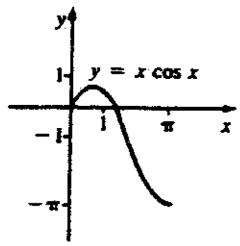
F. Local Maximum and Minimum Values. Since the curve changes from increasing to decreasing close to $x=0.86$ a local maximum exists. Thus $y\doteq 0.86\cos 0.86\doteq 0.56$ is a maximum. Examine the end points of the domain. When $x=0$, $y=0$ and when $x=\pi$, $y=-\pi$.

G. Concavity and Points of Inflection. $y''=-x\cos x-\sin x-\sin x$ $=-x\cos x-2\sin x$. If $-x\cos x-2\sin x=0$, $2\sin x=-x\cos x$ $\Rightarrow \tan x=-\frac{x}{2} \Rightarrow x=0$ and $x\doteq 2.29$.

Interval	y''	y
$0<x<2.29$	$-$	concave down
$2.29<x<\pi$	$+$	concave up

When $x \doteq 2.29$ the point of inflection is approximately $(2.29, -1.51)$

H. Draw the graph.



7. $f'(x) = \sin x(-\sin 3x)3 + \cos 3x \cos x = -3\sin x \sin 3x + \cos x \cos 3x$

$f''(x) = -9\sin x \cos 3x - 3\sin 3x \cos x - 3\cos x \sin 3x - \cos 3x \sin x$

$= -10\sin x \cos 3x - 6\cos x \sin 3x.$

$f''(\frac{\pi}{3}) = -10\sin\frac{\pi}{3}\cos\pi - 6\cos\frac{\pi}{3}\sin\pi = -10 \times \frac{\sqrt{3}}{2} \times (-1) - 6 \times \frac{1}{2} \times 0 = 5\sqrt{3}$

8. (a) $f(x) = \cos x - x$ and $f'(x) = -\sin x - 1$. A sketch of $y = \cos x$ and $y = x$ gives the point of intersection close to $x = 0.7$. Let $x_1 = 0.7$.

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.7 - \frac{f(0.7)}{f'(0.7)} = 0.7 + 0.048237862 = 0.748237862$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.739103454$ $\qquad$ $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.739085133$

$x^5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.739085133$

Therefore $x = 0.739085$ to 6 decimal places.

(b) $f(x) - 2\sin x + x - 2$ and $f'(x) = 2\cos x + 1$. A sketch of $y = 2\sin x$ and $y = 2 - x$ gives the point of intersection close to $x = 0.7$. Let $x_1 = 0.7$.

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.704571569$ $\qquad$ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.704576913$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.704576913$

Therefore $x = 0.704576$ to 6 decimal places.

(c) $f(x) = \sin x - \frac{x}{2}$ and $f'(x) = \cos x - \frac{1}{2}$. Let $x_1 = 1.9$. (Section 6.5, Example 6)

$x_2 = 1.89550594$ $\qquad$ $x_3 = 1.895494267$

$x_4 = 1.895494267$

Therefore $x = 1.895494$ to 6 decimal places.

9. $y = \tan x \Rightarrow y = \frac{\sin x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$

$y = \csc x \Rightarrow y = (\sin x)^{-1} \Rightarrow \frac{dy}{dx} = -(\sin x)^{-2}(\cos x) = -\frac{\cos x}{\sin^2 x}$

$= -\frac{\cos x}{\sin x} \times \frac{1}{\sin x} = -\cot x \csc x$

10. $\sin y + \cos x = 1 \Rightarrow \cos y \frac{dy}{dx} - \sin x = 0 \Rightarrow \frac{dy}{dx} = \frac{\sin x}{\cos y}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos y \cos x + \sin x \sin y \frac{dy}{dx}}{\cos^2 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos y \cos x + \sin x \sin y \frac{\sin x}{\cos y}}{\cos^2 y} = \frac{\cos^2 y \cos x + \sin^2 x \sin y}{\cos^3 y}$$

11. (a) $y = [\sin(x - \sin x)]^{-1} \Rightarrow \frac{dy}{dx} = -[\sin(x - \sin x)]^{-2}\cos(x - \sin x)(1 - \cos x)$

$= \frac{(\cos x - 1)\cos(x - \sin x)}{\sin^2(x - \sin x)} = (\cos x - 1)\cot(x - \sin x)\csc(x - \sin x)$

(b) $y = [\sin x^{\frac{1}{2}}]^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}[\sin x^{\frac{1}{2}}]^{-\frac{1}{2}}(\cos x^{\frac{1}{2}})(\frac{1}{2}x^{-\frac{1}{2}}) = \frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$

(c) $y = (x\cos x)^{\frac{1}{3}} \Rightarrow \frac{dy}{dx} = \frac{1}{3}(x\cos x)^{-\frac{2}{3}}(-x\sin x + \cos x) = \frac{\cos x - x\sin x}{3\sqrt[3]{x^2\cos^2 x}}$

(d) $y' = 3\cos^2(\cos x)[-\sin(\cos x)](-\sin x) + 2\sin(\cos x)\cos(\cos x)(-\sin x)$

$= 3\sin x[\sin(\cos x)][\cos^2(\cos x)] - 2\sin x[\sin(\cos x)][\cos(\cos x)]$

$= \sin x[\sin(\cos x)][\cos(\cos x)][3\cos(\cos x) - 2]$

(e) $y=\sqrt{\cos(\sin^2 x)} \Rightarrow y=[\cos(\sin^2 x)]^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}[\cos(\sin^2 x)]^{-\frac{1}{2}}[-\sin(\sin^2 x)]2\sin x\cos x=\frac{-\sin x\cos x[\sin(\sin^2 x)]}{\sqrt{\cos(\sin^2 x)}}$$

12. $2x\cos 2y\frac{dy}{dx}+\sin 2y=-2y\sin 2x+\cos 2x\frac{dy}{dx}$

$$\Rightarrow (2x\cos 2y-\cos 2x)\frac{dy}{dx}=-\sin 2y-2y\sin 2x \Rightarrow \frac{dy}{dx}=\frac{\sin 2y+2y\sin 2x}{\cos 2x-2x\cos 2y}$$

When $x=\frac{\pi}{4}$, $y=\frac{\pi}{2}$ and the slope of the tangent is $\dfrac{\sin\pi+\pi\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}-\frac{\pi}{2}\cos\pi}=\dfrac{\pi}{\frac{\pi}{2}}=2.$

The equation of the tangent is $y-\frac{\pi}{2}=2(x-\frac{\pi}{4}) \Rightarrow 2y-\pi=4x-\pi$

$\Rightarrow 2x-y=0.$

13. $1+\sec^2(xy)(x\frac{dy}{dx}+y)=\cos y\frac{dy}{dx}-\sin x$

$$\Rightarrow [x\sec^2(xy)-\cos y]\frac{dy}{dx}=-\sin x-y\sec^2(xy)-1$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sin x+y\sec^2(xy)+1}{\cos y-x\sec^2(xy)}$$

EXERCISE 7.3

1. (a) $\frac{dy}{dx} = 3\sec^2 2x(2) = 6\sec^2 2x$

   (b) $\frac{dy}{dx} = -\frac{1}{3}\csc^2 9x(9) = -3\csc^2 9x$

   (c) $\frac{dy}{dx} = 12\sec\frac{1}{4}x\tan\frac{1}{4}x(\frac{1}{4}) = 3\sec\frac{x}{4}\tan\frac{x}{4}$

   (d) $\frac{dy}{dx} = \frac{1}{4}\csc(-8x)\cot(-8x)(-8) = -2\csc 8x\cot 8x$

   (e) $\frac{dy}{dx} = \sec^2 x^2(2x) = 2x\sec^2 x^2$

   (f) $\frac{dy}{dx} = 2\tan x\sec^2 x$

   (g) $\frac{dy}{dx} = \sec\sqrt[3]{x}\tan\sqrt[3]{x}\,(\frac{1}{3}x^{-\frac{2}{3}}) = \frac{\sec\sqrt[3]{x}\tan\sqrt[3]{x}}{3\sqrt[3]{x^2}}$

   (h) $\frac{dy}{dx} = x^2(-\csc x\cot x) + \csc x(2x) = x\csc x(2 - x\cot x)$

   (i) $\frac{dy}{dx} = 3\cot^2(1-2x)^2[-\csc^2(1-2x)^2[2(1-2x)(-2)]$

   $= 12(1-2x)\csc^2(1-2x)^2\cot^2(1-2x)^2$

   (j) $y = 1 + \tan^2 x - \tan^2 x \Rightarrow y = 1 \Rightarrow \frac{dy}{dx} = 0$

   (k) $\frac{dy}{dx} = -\frac{3}{2}(\sec 2x - 1)^{-\frac{5}{2}}\sec 2x\tan 2x(2) = \frac{-3\sec 2x\tan 2x}{\sqrt{(\sec 2x - 1)^5}}$

   (l) $y = \frac{x^2\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \Rightarrow y = x^2\sin x \Rightarrow \frac{dy}{dx} = x^2\cos x + \sin x(2x) = x(x\cos x + 2\sin x)$

   (m) $y = 2x^{\frac{3}{2}} - 2x\cot x \Rightarrow y' = 3\sqrt{x} - 2x(-\csc^2 x) - \cot x(2)$
   $= 3\sqrt{x} + 2x\csc^2 x - 2\cot x$

   (n) $\frac{dy}{dx} = \cos(\tan x)(\sec^2 x) = \sec^2 x\cos(\tan x)$

(0) $\frac{dy}{dx} = 2\tan(\cos x)\sec^2(\cos x)(-\sin x) = -2\sin x\tan(\cos x)\sec^2(\cos x)$

(p) $\frac{dy}{dx} = -3[\tan(x^2-x)^{-2}]^{-4}\sec^2(x^2-x)^{-2}[-2(x^2-x)^{-3}(2x-1)]$

$= \frac{6(2x-1)\sec^2(x^2-x)^{-2}}{(x^2-x)^3\tan^4(x^2-x)^{-2}}$

2. (a) $\sec^2x + \sec y\tan y\frac{dy}{dx} - \frac{dy}{dx} = 0 \Rightarrow (\sec y\tan y - 1)\frac{dy}{dx} = -\sec^2x$

$\Rightarrow \frac{dy}{dx} = \frac{\sec^2x}{1-\sec y\tan y}$

(b) $\sec^2 2x(2) = -\sin 3y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{2\sec^2 2x}{\sin 3y}$

(c) $-\csc^2(x+y)(1+\frac{dy}{dx}) - \csc^2x - \csc^2y\frac{dy}{dx} = 0$

$\Rightarrow -\csc^2(x+y) - \csc^2(x+y)\frac{dy}{dx} - \csc^2x - \csc^2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\csc^2x + \csc^2(x+y)}{\csc^2(x+y) + \csc^2y}$

(d) $2y\frac{dy}{dx} + \csc(xy)\cot(xy)(x\frac{dy}{dx} + y) = 0$

$\Rightarrow 2y\frac{dy}{dx} + x\csc(xy)\cot(xy)\frac{dy}{dx} = -y\csc(xy)\cot(xy)$

$\Rightarrow \frac{dy}{dx} = \frac{-y\csc(xy)\cot(xy)}{2y + x\csc(xy)\cot(xy)}$

(e) $2x + \sec(\frac{x}{y})\tan(\frac{x}{y})\frac{y - x\frac{dy}{dx}}{y^2} = 0$

$\Rightarrow 2xy^2 + y\sec(\frac{x}{y})\tan(\frac{x}{y}) = x\sec(\frac{x}{y})\tan(\frac{x}{y})\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2xy^2 + y\sec(\frac{x}{y})\tan(\frac{x}{y})}{x\sec(\frac{x}{y})\tan(\frac{x}{y})}$

(f) $2y\frac{dy}{dx} = \cos(\tan y)\sec^2 y\frac{dy}{dx} + 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{2y - \sec^2 y\cos(\tan y)}$

Exercise 7.3

3. (a) $\frac{dy}{dx} = 2\cot x(-\csc^2 x) = -2\csc^2 x \cot x$. When $x = \frac{\pi}{4}$, $y = \cot^2\frac{\pi}{4} = 1^2 = 1$.

When $x = \frac{\pi}{4}$ the slope of the tangent is $-2\csc^2\frac{\pi}{4}\cot\frac{\pi}{4} = -2\times 2\times 1 = -4$.

The equation of the tangent is $y - 1 = -4(x - \frac{\pi}{4}) \Rightarrow y - 1 = -4x + \pi$
$\Rightarrow 4x + y - 1 - \pi = 0$.

(b) $\frac{dy}{dx} = \sin x \sec^2\frac{x}{2}(\frac{1}{2}) + \tan\frac{x}{2}\cos x$. When $x = \frac{\pi}{3}$, $y = \sin\frac{\pi}{3}\tan\frac{\pi}{6} = \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{3}} = \frac{1}{2}$.

When $x = \frac{\pi}{3}$ the slope of the tangent is $\frac{1}{2}\sin\frac{\pi}{3}\sec^2\frac{\pi}{6} + \tan\frac{\pi}{6}\cos\frac{\pi}{3}$

$= \frac{1}{2}\times\frac{\sqrt{3}}{2}\times\frac{4}{3} + \frac{1}{\sqrt{3}}\times\frac{1}{2} = \frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$. The equation of the tangent is

$y - \frac{1}{2} = \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) \Rightarrow 6y - 3 = 3\sqrt{3}x - \sqrt{3}\pi \Rightarrow 3\sqrt{3}x - 6y - \sqrt{3}\pi + 3 = 0$

(c) $\frac{dy}{dx} = -\csc 2x \cot 2x(2)$. When $x = -\frac{\pi}{8}$, $y = \csc(-\frac{\pi}{4}) = -\sqrt{2}$.

When $x = -\frac{\pi}{8}$, the slope of the tangent is

$-2\csc(-\frac{\pi}{4})\cot(-\frac{\pi}{4}) = -2\csc\frac{\pi}{4}\cot\frac{\pi}{4} = -2\times\sqrt{2}\times 1 = -2\sqrt{2}$.

The equation of the tangent is $y + \sqrt{2} = -2\sqrt{2}(x + \frac{\pi}{8})$

$\Rightarrow 4y + 4\sqrt{2} = -8\sqrt{2}x - \pi\sqrt{2} \Rightarrow 4y + 8\sqrt{2}x + 4\sqrt{2} + \pi\sqrt{2} = 0$.

(d) $\frac{dy}{dx} = \sec x \tan x - \csc x \cot x$.

When $x = \frac{3\pi}{4}$, $y = \sec\frac{3\pi}{4} + \csc\frac{3\pi}{4} = -\sqrt{2} + \sqrt{2} = 0$.

When $x = \frac{3\pi}{4}$, the slope of the tangent is $\sec\frac{3\pi}{4}\tan\frac{3\pi}{4} - \csc\frac{3\pi}{4}\cot\frac{3\pi}{4}$

$= -\sqrt{2}(-1) - \sqrt{2}(-1) = 2\sqrt{2}$. The equation of the tangent is

$y = 2\sqrt{2}(x - \frac{3\pi}{4}) \Rightarrow y = 2\sqrt{2}x - \frac{3\sqrt{2}\pi}{2} \Rightarrow 4\sqrt{2}x - 2y - 3\sqrt{2}\pi = 0$.

4. $y' = \sec x \tan x + \sec^2 x = \sec x(\sec x + \tan x)$.
For critical numbers $\sec x + \tan x = 0 \Rightarrow \dfrac{1+\sin x}{\cos x} = 0 \Rightarrow \sin x = -1$
or $\cos x = 0$, $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow$ no solution. Therefore the curve is always increasing or decreasing in the given interval. Test a point:
if $x = \frac{\pi}{4}$, $y' = \sqrt{2}(\sqrt{2}+1) > 0 \Rightarrow$ the curve is always increasing.

5. (a) $y = \csc x - \cot x \Rightarrow y = \dfrac{1-\cos x}{\sin x}$. For vertical asymptotes $\sin x = 0$, $0 < x < \pi$
$\Rightarrow$ no solution within the domain but $x = 0$, or $x = \pi$ may be vertical asymptotes.
$\lim_{x \to 0^+} y = 0$ and $\lim_{x \to \pi^-} y = +\infty$. Therefore $x = \pi$ is the vertical asymptote

(b) $y = \sin x - \tan x \Rightarrow y = \dfrac{\sin x \cos x - \sin x}{\cos x}$. For vertical asymptotes $\cos x = 0$,
$-\frac{\pi}{2} < x < \frac{3\pi}{2} \Rightarrow x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$. $\lim_{x \to -\frac{\pi}{2}^+} y = +\infty$, $\lim_{x \to \frac{\pi}{2}^-} y = -\infty$, $\lim_{x \to \frac{\pi}{2}^+} y = +\infty$
and $\lim_{x \to \frac{3\pi}{2}^-} y = -\infty$

6. $\dfrac{dy}{dx} = -\csc x \cot x + \csc^2 x = \dfrac{-\cos x + 1}{\sin^2 x}$. For critical numbers $\cos x = 1$
or $\sin x = 0$, $0 < x < \pi \Rightarrow$ no solution. If $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 1 > 0$ and the curve is always increasing. As $x \to 0^+, y \to 0$ and as $x \to \pi^-$, $y \to \infty$.

7. $\dfrac{dy}{dx} = \cos x - \sec^2 x$ and $\dfrac{d^2y}{dx^2} = -\sin x - 2\sec^2 x \tan x = -\sin x - \dfrac{2\sin x}{\cos^3 x}$
$= -\dfrac{\sin x(\cos^3 x + 2)}{\cos^3 x}$. Possible points of inflection occur when $\sin x = 0$ and
vertical asymptotes occur when $\cos x = 0$, $-\frac{\pi}{2} < x < \frac{3\pi}{2} \to x = 0, \frac{\pi}{2}, \pi$

Interval	$-\sin x$	$\cos^3 x$	$\cos^3 x + 2$	$f''(x)$	$f(x)$
$-\frac{\pi}{2} < x < 0$	+	+	+	+	concave up
$0 < x < \frac{\pi}{2}$	−	+	+	−	concave down
$\frac{\pi}{2} < x < \pi$	−	−	+	+	concave up
$\pi < x < \frac{3\pi}{2}$	+	−	+	−	concave down

8. A. Domain. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

B. Intercepts. $f(0)=0$ is the y-intercept. x-intercepts occur when $x\tan x = 0$ $\Rightarrow x = 0$.

C. Symmetry. $f(-x) = -x\tan(-x) = x\tan x = f(x)$ $\Rightarrow$ symmetry about the y-axis.

D. Asymptotes. For vertical asymptotes set $\cos x = 0 \Rightarrow x = -\frac{\pi}{2}, \frac{\pi}{2}$.
$\lim_{x\to -\frac{\pi}{2}^+} y = \infty$ and $\lim_{x\to \frac{\pi}{2}^-} y = \infty$. Therefore vertical asymptotes are $x = \pm\frac{\pi}{2}$.

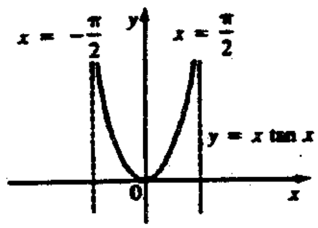
E. Intervals of Increase and Decrease. $\frac{dy}{dx} = x\sec^2 x + \tan x = \frac{x + \sin x\cos x}{\cos^2 x}$.
Critical numbers occur when $x + \sin x\cos x = 0$
$\Rightarrow \sin 2x = -2x,\ -\pi < 2x < \pi \Rightarrow x = 0$ or when $\sin x = 0 \Rightarrow x = 0$.

Interval	$x+\sin x\cos x$	$\cos^2 x$	$f'(x)$	$f(x)$
$-\frac{\pi}{2} < x < 0$	$-$	$+$	$-$	decreasing
$0 < x < \frac{\pi}{2}$	$+$	$+$	$+$	increasing

F. Local Maximum and Minimum values. When $x = 0$, minimum value of $y = 0$.

G. Concavity and Points of Inflection. $y'' = 2x\sec^2 x\tan x + 2\sec^2 x$
$= 2\sec^2 x(x\tan x + 1)$. If $2\sec^2 x(x\tan x + 1) = 0$, $\tan x = -\frac{1}{x}$ has no solution in our domain. There are no points of inflection. The curve is concave up.

H. Draw the Graph.



9. (a) $\frac{d}{dx}\sec x = \frac{d}{dx}(\cos x)^{-1} = -(\cos x)^{-2}(-\sin x) = \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x\tan x$

(b) $\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{\sin x(-\sin x) - \cos x\cos x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$

$= -\frac{1}{\sin^2 x} = -\csc^2 x.$

Exercise 7.3

10. $f'(x) = -2\csc^2 2x$ and $f''(x) = -4\csc 2x(-\csc 2x \cot 2x)(2)$

$f(x) = f''(x) \Rightarrow \cot 2x = 8\csc^2 2x \cot 2x \Rightarrow \cot 2x(1 - 8\csc^2 2x) = 0 \Rightarrow \cot 2x = 0$

$\Rightarrow \cos 2x = 0,\ 0 \le 2x \le 4\pi \Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$

11. $2x + 2\tan y \sec^2 y \frac{dy}{dx} = 2\sec^2 y \tan y \frac{dy}{dx} - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -2x.$

$2x\frac{dx}{dy} + 2\tan y \sec^2 y = 2\sec^2 y \tan y - 1 \Rightarrow \frac{dx}{dy} = -\frac{1}{2x}.$

Therefore $2x = \frac{1}{2x} \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}.$

12. $f(x) = \sqrt{\sec^3(\sqrt[4]{x})} \Rightarrow f(x) = [\sec x^{\frac{1}{4}}]^{\frac{3}{2}} \Rightarrow f'(x) = \frac{3}{2}(\sec x^{\frac{1}{4}})^{\frac{1}{2}} \sec x^{\frac{1}{4}} \tan x^{\frac{1}{4}}(\frac{1}{4}x^{-\frac{3}{4}})$

$= \frac{3\sec^{\frac{3}{2}}(\sqrt[4]{x})\tan(\sqrt[4]{x})}{8(\sqrt[4]{x^3})}.$

13. $x = \cos 3t \Rightarrow \frac{dx}{dt} = -3\sin 3t$ and $y = \sin^2 3t \Rightarrow \frac{dy}{dt} = 6\sin 3t\cos 3t.$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{dy}{dt} \div \frac{dx}{dt} = \frac{6\sin 3t \cos 3t}{-3\sin 3t} = -2\cos 3t = -2x$ and $\frac{d^2y}{dx^2} = -2.$

EXERCISE 7.4

1. (a) $f'(x) = 1 - 2\cos x$. $f'(x) = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$. $f''(x) = 2\sin x$.
$f''(\frac{\pi}{3}) = \sqrt{3} > 0 \Rightarrow f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3} \doteq -0.685$ is a local minimum.

$f''(\frac{5\pi}{3}) = -\sqrt{3} < 0 \Rightarrow f(\frac{5\pi}{3}) = \frac{5\pi}{3} + \sqrt{3} \doteq 6.968$ is a local maximim.

(b) $f'(x) = 1 - \sin x \geq 0$. $f'(x) = 0 \Rightarrow \sin x = 1 \Rightarrow x = \frac{\pi}{2}$. $f''(x) = -\cos x$
$f''(\frac{\pi}{2}) = 0 \Rightarrow$ use first derivative test.

Interval	$f'(x)$	$f(x)$
$0 < x < \frac{\pi}{2}$	+	increasing
$\frac{\pi}{2} < x < 2\pi$	+	increasing

There are no local maxima or minima.

(c) $f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x = 4\sin x\cos x(\sin^2 x - \cos^2 x)$
$= -2\sin 2x\cos 2x$. $f'(x) = 0 \Rightarrow \sin 2x = 0$ or $\cos 2x = 0$, $0 \leq 2x \leq 4\pi$
$\Rightarrow 2x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \frac{7\pi}{2}, 4\pi \Rightarrow x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$.

Interval	-2	$\sin 2x$	$\cos 2x$	$f'(x)$	$f(x)$
$0 < x < \frac{\pi}{4}$	−	+	+	−	decreasing
$\frac{\pi}{4} < x < \frac{\pi}{2}$	−	+	−	+	increasing
$\frac{\pi}{2} < x < \frac{3\pi}{4}$	−	−	−	−	decreasing
$\frac{3\pi}{4} < x < \pi$	−	−	+	+	increasing
$\pi < x < \frac{5\pi}{4}$	−	+	+	−	decreasing
$\frac{5\pi}{4} < x < \frac{3\pi}{2}$	−	+	−	+	increasing
$\frac{3\pi}{2} < x < \frac{7\pi}{4}$	−	−	−	−	decreasing
$\frac{7\pi}{4} < x < 2\pi$	−	+	−	+	increasing

The local minima are $f(\frac{\pi}{4}) = f(\frac{3\pi}{4}) = f(\frac{5\pi}{4}) = f(\frac{7\pi}{4}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
The local maxima are $f(0) = f(\frac{\pi}{2}) = f(\pi) = f(\frac{3\pi}{2}) = f(2\pi) = 1$.

(d) $f'(x)=x\cos x+\sin x-\sin x=x\cos x$. $f'(x)=0 \Rightarrow x=0$ or $\cos x=0$
$\Rightarrow x=-\frac{\pi}{2}, 0, \frac{\pi}{2}$. $f''(x)=-x\sin x+\cos x$.
$f''(-\frac{\pi}{2})=-\frac{\pi}{2}<0 \Rightarrow f(-\frac{\pi}{2})=\frac{\pi}{2}$ is a local maximum.
$f''(0)=1>0 \Rightarrow f(0)=1$ is a local minimum.
$f''(\frac{\pi}{2})=-\frac{\pi}{2}<0 \Rightarrow f(\frac{\pi}{2})=\frac{\pi}{2}$ is a local maximum.

2. (a) $f'(t)=2\cos t+2\cos 2t$. $f'(t)=0 \Rightarrow 2\cos t+4\cos^2 t-2=0$
$\Rightarrow 2\cos^2 t+\cos t-1=0 \Rightarrow (2\cos t-1)(\cos t+1)=0 \Rightarrow \cos t=\frac{1}{2}$
or $\cos t=-1 \Rightarrow t=-\pi, -\frac{\pi}{3}, \frac{\pi}{3}, \pi$.

Interval	$2\cos t-1$	$\cos t+1$	$f'(t)$	$f(t)$
$-\pi<x<-\frac{\pi}{3}$	$-$	$+$	$-$	decreasing
$-\frac{\pi}{3}<x<\frac{\pi}{3}$	$+$	$+$	$+$	increasing
$\frac{\pi}{3}<x<\pi$	$-$	$+$	$-$	decreasing

$f(-\frac{\pi}{3})=-\frac{3\sqrt{3}}{2}$ is a local minimum. $f(\frac{\pi}{3})=\frac{3\sqrt{3}}{2}$ is a local maximum.
Test the end points of the domain. $f(-\pi)=-1$ and $f(\pi)=-1$.

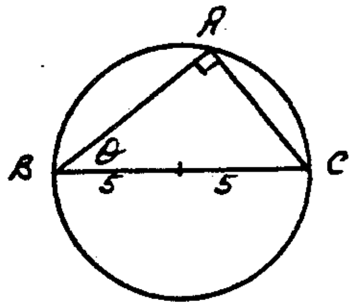
Therefore $-\frac{3\sqrt{3}}{2}$ is an absolute minimum and $\frac{3\sqrt{3}}{2}$ is an absolute maximum.

(b) $f'(t)=2\sin t\cos t+4\cos t\sin t=6\sin t\cos t$. $f'(t)=0 \Rightarrow \sin t=0$ or $\cos t=0$
$\Rightarrow t=-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$.

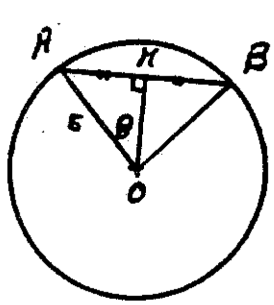
Interval	$\sin t$	$\cos t$	$f'(t)$	$f(t)$
$-\pi<x<-\frac{\pi}{2}$	$-$	$-$	$+$	increasing
$-\frac{\pi}{2}<x<0$	$-$	$+$	$-$	decreasing
$0<x<\frac{\pi}{2}$	$+$	$+$	$+$	increasing
$\frac{\pi}{2}<x<\pi$	$+$	$-$	$-$	decreasing

$f(-\frac{\pi}{2})=1$ is a local maximum, $f(0)=-2$ is a local minimum and $f(\frac{\pi}{2})=1$ is a local maximum. Test the end points of the domain.
$f(-\pi)=-2$ and $f(\pi)=-2$.
Therefore -2 is an absolute minimum and 1 is an absolute maxi

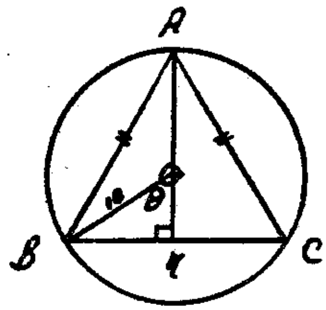
3. Let the area be A cm^2 and $\angle B = \theta$.
Since $A = \frac{1}{2} AB \times AC$,
and $\angle BAC = \frac{\pi}{2}$ (angle in a semi-circle),
and $AB = 10\cos\theta$ and $AC = 10\sin\theta$, $0 < \theta < \frac{\pi}{2}$,
therefore $A = 50\sin\theta\cos\theta = 25\sin 2\theta$.
$A' = 50\cos 2\theta$. $A' = 0 \Rightarrow \theta = \frac{\pi}{4}$.
When $\theta = \frac{\pi}{4}$, $A'' = -100\sin 2\theta = -100 \times 1 < 0 \Rightarrow$ a maximum.
Therefore $\theta = \frac{\pi}{4}$ produces the triangle of maximum area.



4. Since $\triangle AOB$ is isosceles the altitude OM bisects the base and $\angle AOB$.
Let the area of $\triangle AOB = A$ cm^2
and $\angle AOM = \theta$, $0 < \theta < \frac{\pi}{2}$.
Since $A = OM \times MA$,
and $OM = 5\cos\theta$, and $MA = 5\sin\theta$,
therefore $A = 25\sin\theta\cos\theta = \frac{25}{2}\sin 2\theta$.
$A' = 25\cos 2\theta$. $A' = 0 \Rightarrow \theta = \frac{\pi}{4}$.
When $\theta = \frac{\pi}{4}$, $A'' = -50\cos 2\theta = -50 \times 1 < 0 \Rightarrow$ a maximum.
Therefore $\angle AOB = 2\theta = \frac{\pi}{2}$ produces the triangle of maximum area.

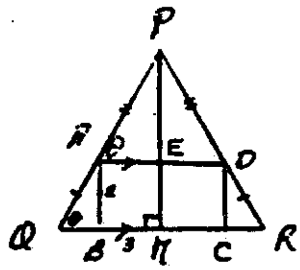


5. Since $\triangle BAC$ is isosceles the altitude AM passes through O, bisects the base, and bisects $\angle BAC$.
Let the area of $\triangle BAC = A$ cm^2
and $\angle BAC = \theta$, $0 < \theta < \pi$.
Therefore $\angle BOM = 2\angle BAM = \theta$.
Since $A = AM \times BM$,
and $BM = 10\sin\theta$, and $AM = AO + OM = 10 + 10\cos\theta$,
therefore $A = 10\sin\theta(10 + 10\cos\theta) = 100\sin\theta + 100\sin\theta\cos\theta$.
$A' = 100\cos\theta - 100\sin^2\theta + 100\cos^2\theta = 100(2\cos^2\theta + \cos\theta - 1)$.
$A' = 0 \Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$.
$A'' = -100\sin\theta - 200\sin\theta\cos\theta - 200\sin\theta\cos\theta = -100(\sin\theta + \sin 2\theta)$.
When $\theta = \frac{\pi}{3}$, $\sin\theta + \sin 2\theta > 0$ and $A'' < 0 \Rightarrow$ maximum.
Therefore $\angle BAC = \frac{\pi}{3}$ produces the triangle of maximum area.



6. Since ΔQPR is isosceles the altitude PM bisects the base.

Let the area of $\Delta QPR = A$ cm^2 and $\angle PQR = \theta$, $0 < \theta < \frac{\pi}{2}$.



Since $A = QM \times MP$,

and $QM = QB + BM = 2\cot\theta + 3$,

and $PM = ME + EP = 2 + 3\tan\theta$,

therefore $A = (3 + 2\cot\theta)(2 + 3\tan\theta) = 6 + 9\tan\theta + 4\cot\theta + 6$.

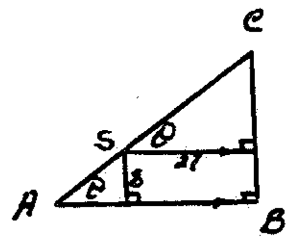
$A' = 9\sec^2\theta - 4\csc^2\theta = (3\sec\theta - 2\csc\theta)(3\sec\theta + 2\csc\theta)$.

$A' = 0 \Rightarrow 3\sec\theta - 2\csc\theta = 0 \Rightarrow \tan\theta = \frac{2}{3} \Rightarrow \theta = 0.58800$ radians.

$A'' = 18\sec^2\theta\tan\theta + 8\csc^2\theta\cot\theta > 0$ since θ is acute $\Rightarrow$ minimum.

Therefore $\angle PQR = 0.58800$ radians produces the triangle of minimum area.

7. The length of the ladder is AC m and the angle the ladder makes with the ground is $\angle CAB = \theta$, $0 < \theta < \frac{\pi}{2}$.



$AC = AS + SC = 8\csc\theta + 27\sec\theta$.

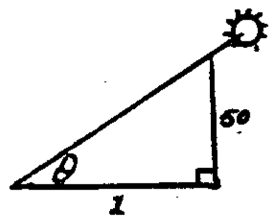
$AC' = -8\csc\theta\cot\theta + 27\sec\theta\tan\theta$.

$AC' = 0 \Rightarrow \frac{\sec\theta\tan\theta}{\csc\theta\cot\theta} = \frac{8}{27} \Rightarrow \tan^3\theta = \frac{8}{27} \Rightarrow \tan\theta = \frac{2}{3} \Rightarrow \theta = 0.58800$ radians.

$AC'' = 8\csc^3\theta + 8\csc\theta\cot^2\theta + 27\sec^3\theta + 27\sec\theta\tan^2\theta > 0$ since θ is acute $\Rightarrow$ min.

Therefore $\angle CAB = 0.58800$ radians produces the ladder of shortest length.

8. Let the length of the shadow be x m and the angle of elevation of the sun be θ. We are given $\frac{d\theta}{dt} = -\frac{1}{4}$.



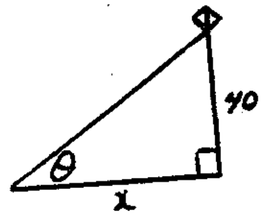
Since $\frac{x}{50} = \cot\theta$, $\frac{1}{50}\frac{dx}{dt} = -\csc^2\theta\frac{d\theta}{dt}$.

Therefore $\frac{dx}{dt} = -50\csc^2\theta\frac{d\theta}{dt}$. When $\theta = \frac{\pi}{4}$, $\frac{dx}{dt} = -50 \times 2 \times -\frac{1}{4} = 25$ m/h.

Therefore the shadow is lengthening at 25 m/h when the angle of elevation of the sun is $\frac{\pi}{4}$.

Exercise 7.4

9. Let the projection of the string on the ground be x m and the angle it makes with the ground be θ.



We are given $\frac{dx}{dt} = 3$ m/s.

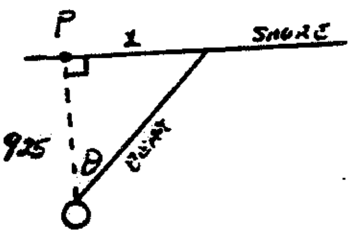
Since $\frac{x}{40} = \cot\theta$, $\frac{1}{40}\frac{dx}{dt} = -\csc^2\theta\frac{d\theta}{dt}$.

Therefore $\frac{d\theta}{dt} = -\frac{1}{40\csc^2\theta}\frac{dx}{dt}$. When the length of the string is 80 m,

$$\frac{d\theta}{dt} = -\frac{1}{40 \times \left(\frac{80}{40}\right)^2} \times 3 = -\frac{40 \times 3}{6400} = -0.01875 \text{ m/s.}$$

Therefore the angle between the string and the ground is decreasing at 0.02 m/s when the length of the string is 80 m.

10. Let the distance along the shore, measured from point P, be x m and the angle between the beam and the perpendicular be θ.



We are given $\frac{d\theta}{dt} = 4\pi$ rad/min.

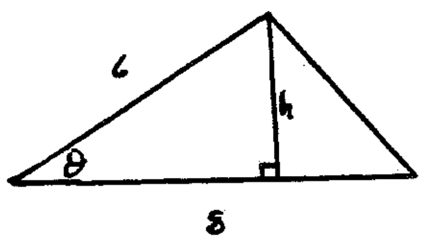
Since $\frac{x}{925} = \tan\theta$, $\frac{dx}{dt} = 925\sec^2\theta\frac{d\theta}{dt}$.

When $x = 0$, $\frac{dx}{dt} = 925(1)^2(4\pi) \doteq 11\,624$ m/min.

When $x = 1275$, $\sec\theta \doteq \frac{1575.1984}{925}$ and $\frac{dx}{dt} = 925 \times \frac{2481250}{925^2} \times 4\pi \doteq 33\,708.44009$ m/min.

Therefore at the closest point the beam sweeps along the shore at 11 624 m/min and at a point 1275 m from the closest point it is moving at 33 708 m/min.

11. Let the height of the triangle be h m, the angle between the given sides be θ, and the area be A m^2.



We are given $\frac{d\theta}{dt} = -0.035$ rad/s.

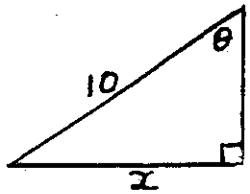
Since $A = 4h = 4(6\sin\theta) = 24\sin\theta$, $\frac{dA}{dt} = 24\cos\theta\frac{d\theta}{dt}$.

When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = 24 \times \frac{\sqrt{3}}{2} \times -0.035 = -0.727$ m/min.

Therefore the area is decreasing at 0.727 m^2/min.

Exercise 7.4

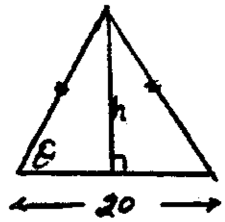
12. Let the angle between the ladder and the top of the wall be θ and the distance from the foot of the ladder to the wall be x m. We are given $\frac{dx}{dt} = 2$ m/s.



Since $\frac{x}{10} = \sin\theta$, $\frac{dx}{dt} = 10\cos\theta\frac{d\theta}{dt}$, and $\frac{d\theta}{dt} = \frac{1}{10\cos\theta}\frac{dx}{dt}$.

When $\theta = \frac{\pi}{4}$, $\frac{d\theta}{dt} = \frac{1}{10 \times \frac{1}{\sqrt{2}}} \times 2 = \frac{\sqrt{2}}{5}$. Therefore the angle is changing at the rate of $\frac{\sqrt{2}}{5}$ rad/s when the angle is $\frac{\pi}{4}$.

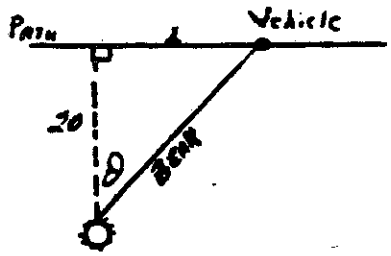
13. Let the altitude be h cm and the base angle be θ. We are given $\frac{dh}{dt} = 1$ cm/min.



Since $\frac{h}{10} = \tan\theta$, $\frac{dh}{dt} = 10\sec^2\theta\frac{d\theta}{dt}$ and $\frac{d\theta}{dt} = \frac{1}{10\sec^2\theta}\frac{dh}{dt}$.

When the area of the triangle is 100 , $h = 10$, and the equal sides are $10\sqrt{2}$. Therefore $\frac{d\theta}{dt} = \frac{1}{10 \times 2} = \frac{1}{20}$. Therefore the base angle is increasing at 0.05 rad/s when the area is 100 cm^2.

14. Let the distance from the point on the path closest to the searchlight and the vehicle be x m. Let the angle between the beam of the searchlight and the perpendicular to the path be θ. We are given $\frac{dx}{dt} = 4$ m/s.

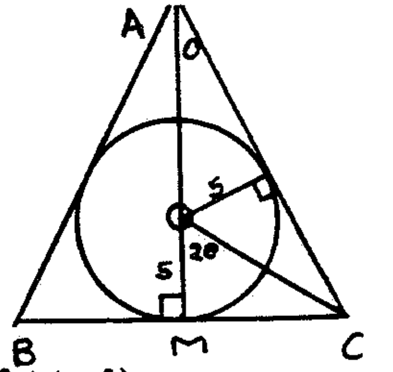


Since $\frac{x}{20} = \tan\theta$, $\frac{dx}{dt} = 20\sec^2\theta\frac{d\theta}{dt}$ and $\frac{d\theta}{dt} = \frac{1}{20\sec^2\theta}\frac{dx}{dt}$.

When $x = 15$, the length of the beam is 25, and $\frac{d\theta}{dt} = \frac{1}{20 \times \frac{25}{16}} \times 4 = \frac{16}{125}$.

Therefore the searchlight is rotating at 0.128 rad/s at the required time.

Exercise 7.4



15. Let $\angle BAC = 2\theta$. Since $\triangle ABC$ is isosceles, AM the perpendicular to BC passes through point O, and $\angle MAC = \theta$, and $\angle MOC = 2\theta$. Let the area of $\triangle ABC = A$.

$A = AM \times MC = (AO + OM)(MC)$

$= (5\csc\theta + 5)[(5\csc\theta + 5)\tan\theta] = 25(\csc\theta + 1)(\sec\theta + \tan\theta)$

$= 25(\csc\theta\sec\theta + \sec\theta + \sec\theta + \tan\theta) = 25(\csc\theta\sec\theta + 2\sec\theta + \tan\theta)$

$A' = 25(\csc\theta\sec\theta\tan\theta - \sec\theta\csc\theta\cot\theta + 2\sec\theta\tan\theta + \sec^2\theta)$.

$A' = 0 \Rightarrow \dfrac{1}{\cos^2\theta} - \dfrac{1}{\sin^2\theta} + \dfrac{2\sin\theta}{\cos^2\theta} + \dfrac{1}{\cos^2\theta} = 0$

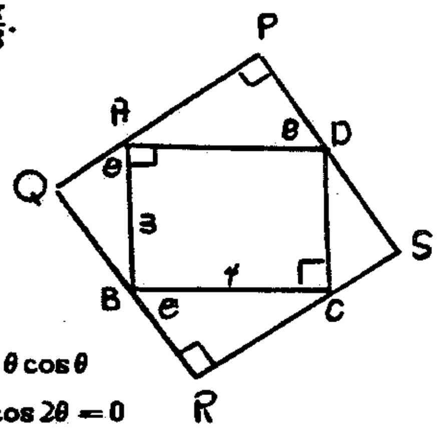
$\Rightarrow \dfrac{\sin^2\theta - \cos^2\theta + 2\sin^3\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta} = 0 \Rightarrow \sin^2\theta - 1 + \sin^2\theta + 2\sin^3\theta + \sin^2\theta = 0$

$\Rightarrow 2\sin^3\theta + 3\sin^2\theta - 1 = 0 \Rightarrow (\sin\theta + 1)(2\sin^2\theta + \sin\theta - 1) = 0$

$\Rightarrow (\sin\theta + 1)(2\sin\theta - 1)(\sin\theta + 1) = 0 \Rightarrow 2\sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

If $\theta = 0.5$, $A' < 0$ and if $\theta = 0.6$ $A' > 0$

$\Rightarrow$ a minimum area occurs when $\angle BAC = 2\theta = \frac{\pi}{3}$.



16. Let $\angle QAB = \theta$, $0 < \theta < \frac{\pi}{2}$.

Therefore $\angle PDA = \angle RBC = \theta$.

Let the area of rectangle PQRS be A m^2.

Since $A = QP \times QR = (QA + AP)(QB + BR)$,

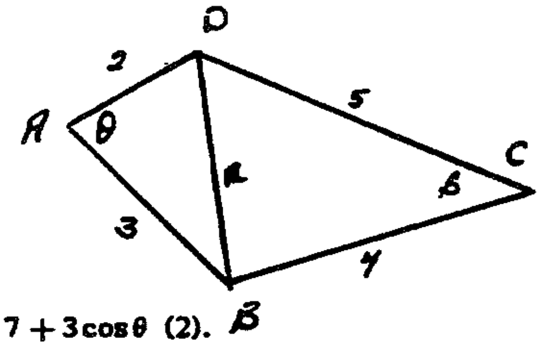
therefore $A = (3\cos\theta + 4\sin\theta)(3\sin\theta + 4\cos\theta)$

$\Rightarrow A = 9\sin\theta\cos\theta + 12\cos^2\theta + 12\sin^2\theta + 16\sin\theta\cos\theta$

$\Rightarrow A = \frac{25}{2}\sin 2\theta + 12$. $A' = 25\cos 2\theta$. $A' = 0 \Rightarrow \cos 2\theta = 0$

$\Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$. $A'' = -50\sin 2\theta < 0$ when $\theta = \frac{\pi}{4} \Rightarrow$ maximum.

Therefore the maximum area of PQRS $= \frac{7}{\sqrt{2}} \times \frac{7}{\sqrt{2}} = \frac{49}{2} = 24.5\ m^2$.



17. Let A = the area of quadrilateral ABCD and m = the diagonal DB. Let $\angle DAB = \theta$ and $\angle DCB = \beta$. Since $A = \triangle DAB + \triangle DBC$,

therefore $A = \frac{1}{2} \times 2 \times 3\sin\theta + \frac{1}{2} \times 4 \times 5\sin\beta$

$\Rightarrow A = 3\sin\theta + 5\cos\beta$ (1).

Since $m^2 = 4 + 9 - 12\cos\theta = 16 + 25 - 40\cos\beta$,

therefore $40\cos\beta = 28 + 12\cos\theta \Rightarrow 10\cos\beta = 7 + 3\cos\theta$ (2).

Squaring (2) we get $100\cos^2\beta = (7 + 3\cos\theta)^2 \Rightarrow 1 - \sin^2\beta = \frac{1}{100}(7 + 3\cos\theta)^2$

$$\Rightarrow \sin^2\beta = \frac{100-(7+3\cos\theta)^2}{100},\ \sin\beta > 0 \ \Rightarrow\ \sin\beta = \frac{1}{10}[100-(7+3\cos\theta)^2]^{\frac{1}{2}}$$

substituting in (1) we get $A = 3\sin\theta + [100-(7+3\cos\theta)^2]^{\frac{1}{2}}$.

$$A' = 3\cos\theta + \frac{-2(7+3\cos\theta)(-3\sin\theta)}{2[100-(7+3\cos\theta)^2]^{\frac{1}{2}}} = 0$$

$$A' = 0 \ \Rightarrow \text{ if } 3\cos\theta = \frac{3\sin\theta(7+3\cos\theta)}{[100-(7+3\cos\theta)^2]^{\frac{1}{2}}}$$

$$\Rightarrow\ 9\cos^2\theta = \frac{9\sin^2\theta(7+3\cos\theta)^2}{100-(7+3\cos\theta)^2}$$

$$\Rightarrow\ 900\cos^2\theta - 9\cos^2\theta(7+3\cos\theta)^2 = 9\sin^2\theta(7+3\cos\theta)^2$$

$$\Rightarrow\ 900\cos^2\theta = 9(7+3\cos\theta)^2 \ \Rightarrow\ 30\cos\theta = \pm 3(7+3\cos\theta)$$

therefore $30\cos\theta = 21 + 9\cos\theta \ \Rightarrow\ \cos\theta = 1 \ \Rightarrow\ \theta = 0$ (inadmissible)

or $30\cos\theta = -21 - 9\cos\theta \ \Rightarrow\ 39\cos\theta = -21 \ \Rightarrow\ \cos\theta = -\frac{7}{13}$.

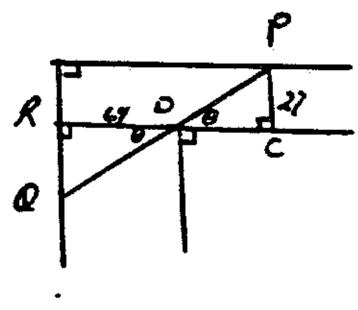
Substituting in (1) we get $10\cos\beta = 7 - \frac{21}{13} \ \Rightarrow\ \cos\beta = \frac{7}{13}$.

Since $\cos\theta + \cos\beta = 0$, $2\cos\frac{\theta+\beta}{2}\cos\frac{\theta-\beta}{2} = 0$.

Therefore $\frac{\theta+\beta}{2} = \frac{\pi}{2}$ or $\frac{\theta-\beta}{2} = \frac{\pi}{2} \ \Rightarrow\ \theta+\beta = \pi$ or $\theta - \beta = \pi$ (inadmissible)

Since $\theta + \beta = \pi$ the opposite angles are supplementary.

18. This is a minimum question despite the request for a maximum. We want the shortest line passing through D from one wall to the other wall.



Let $\angle PDC = \theta, 0 < \theta < \frac{\pi}{2}$. Therefore $\angle RDQ = \theta$.

The length of the beam is $PQ = PD + DQ$.

$PQ = 27\csc\theta + 64\sec\theta$.

$PQ' = -27\csc\theta\cot\theta + 64\sec\theta\tan\theta$.

$PQ' = 0 \ \Rightarrow\ 64\frac{\sin\theta}{\cos^2\theta} = 27\frac{\cos\theta}{\sin^2\theta} \ \Rightarrow\ \frac{\sin^3\theta}{\cos^3\theta} = \frac{27}{64} \ \Rightarrow\ \frac{\sin\theta}{\cos\theta} = \frac{3}{4} \ \Rightarrow\ \tan\theta = \frac{3}{4}$.

Therefore $PQ = 27\left(\frac{5}{3}\right) + 64\left(\frac{5}{4}\right) = 45 + 80 = 125$ m.

$PQ'' = 27\csc\theta\cot^2\theta + 27\csc^3\theta + 64\sec\theta\tan^2\theta + 64\sec^3\theta > 0$ in $\left(0, \frac{\pi}{2}\right)$. $\Rightarrow$ minimum.

The longest such beam is 125 m.

EXERCISE 7.5

1. (a) Let $y=\sin^{-1}\frac{1}{2} \Rightarrow \sin y=\frac{1}{2}, -\frac{\pi}{2}\le y\le\frac{\pi}{2} \Rightarrow y=\frac{\pi}{6}$

(b) Let $y=\cos^{-1}(-\frac{1}{2}) \Rightarrow \cos y=-\frac{1}{2}, 0\le y\le\pi \Rightarrow y=\frac{2\pi}{3}$

(c) Let $y=\tan^{-1}(-1) \Rightarrow \tan y=-1, -\frac{\pi}{2}<x<\frac{\pi}{2} \Rightarrow y=-\frac{\pi}{4}$

(d) Let $y=\sin^{-1}(-\frac{1}{\sqrt{2}}) \Rightarrow \sin y=-\frac{1}{\sqrt{2}}, -\frac{\pi}{2}\le y\le\frac{\pi}{2} \Rightarrow y=-\frac{\pi}{4}$

(e) Let $y=\cos^{-1}\frac{1}{\sqrt{2}} \Rightarrow \cos y=\frac{1}{\sqrt{2}}, 0\le y\le\pi \Rightarrow y=\frac{\pi}{4}$

(f) Let $y=\tan^{-1}\frac{\sqrt{3}}{3} \Rightarrow \tan y=\frac{1}{\sqrt{3}}, -\frac{\pi}{2}<y<\frac{\pi}{2} \Rightarrow y=\frac{\pi}{6}$

2. (a) Let $y=\tan^{-1}\frac{12}{13} \Rightarrow \tan y=\frac{12}{13}, 0<y<\frac{\pi}{2} \Rightarrow \sin y=\frac{12}{\sqrt{313}}$

(b) Let $y=\sin^{-1}\frac{4}{5} \Rightarrow \sin y=\frac{4}{5}, 0\le y\le\frac{\pi}{2} \Rightarrow \cos y=\frac{3}{5}$

(c) Let $y=\cos^{-1}(-\frac{1}{3}) \Rightarrow \cos y=-\frac{1}{3}, \frac{\pi}{2}\le y\le\pi \Rightarrow \tan y=-2\sqrt{2}$

(d) Let $y=\cos^{-1}\frac{7}{8} \Rightarrow \cos y=\frac{7}{8}, 0\le y\le\frac{\pi}{2} \Rightarrow \sin y=\frac{\sqrt{15}}{8}$

(e) Let $y=\tan^{-1}\frac{7}{5} \Rightarrow \tan y=\frac{7}{5}, 0<y<\frac{\pi}{2} \Rightarrow \cos y=\frac{5}{\sqrt{74}}$

(f) Let $y=\sin^{-1}(-\frac{2}{\sqrt{5}}) \Rightarrow \sin y=-\frac{2}{\sqrt{5}}, -\frac{\pi}{2}\le y\le 0 \Rightarrow \tan y=-2$

3. (a) $\sin(\sin^{-1}\frac{\sqrt{3}}{2})=\frac{\sqrt{3}}{2}$

(b) $\sin^{-1}(\sin\frac{3\pi}{7})=\frac{3\pi}{7}$

(c) $\sin^{-1}(\sin\frac{3\pi}{4})=\sin^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{4}$

(d) $\cos(\cos^{-1}\frac{\sqrt{2}}{2})=\frac{\sqrt{2}}{2}$

(e) $\cos^{-1}(\cos\frac{7\pi}{8})=\frac{7\pi}{8}$

(f) $\cos^{-1}\left[\cos(-\frac{\pi}{4})\right]=\cos^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{4}$

(g) $\tan(\tan^{-1}\sqrt{3})=\sqrt{3}$

(h) $\tan^{-1}(\tan\frac{5\pi}{6})=\tan^{-1}(-\frac{1}{\sqrt{3}})=-\frac{\pi}{6}$

(i) $\tan^{-1}\left[\tan(-\frac{\pi}{6})\right]=-\frac{\pi}{6}$

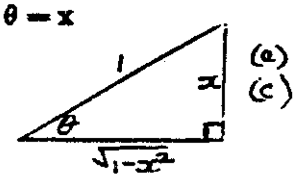
Exercise 7.5

4. (a) Let $\theta = \sin^{-1}x \Rightarrow \sin\theta = x$

$\Rightarrow \cos\theta = \sqrt{1-x^2}$

(b) Let $\theta = \cos^{-1}x \Rightarrow \cos\theta = x$

$\Rightarrow \sin\theta = \sqrt{1-x^2}$

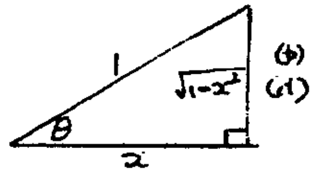


(c) Let $\theta = \sin^{-1}x \Rightarrow \sin\theta = x$

$\Rightarrow \tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

(d) Let $\theta = \cos^{-1}x \Rightarrow \cos\theta = x$

$\Rightarrow \tan\theta = \dfrac{\sqrt{1-x^2}}{x}$

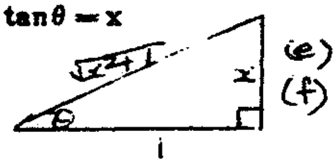


(e) Let $\theta = \tan^{-1}x \Rightarrow \tan\theta = x$

$\Rightarrow \sin\theta = \dfrac{x}{\sqrt{1+x^2}}$

(f) Let $\theta = \tan^{-1}x \Rightarrow \tan\theta = x$

$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{1+x^2}}$



(g) Let $\theta = \sin^{-1}x \Rightarrow x = \sin\theta$. Now $\cos(2\sin^{-1}x) = \cos 2\theta = 1 - 2\sin^2\theta = 1 - 2x^2$

5. (a) Let $\theta = \sin^{-1}\frac{3}{5} \Rightarrow \sin\theta = \frac{3}{5}$. Now $\sin(2\sin^{-1}\frac{3}{5})$
$= \sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

(b) Let $\theta = \sin^{-1}\frac{5}{13} \Rightarrow \sin\theta = \frac{5}{13}$. Now $\cos(2\sin^{-1}\frac{5}{13}) = \cos 2\theta = 1 - 2\sin^2\theta$
$= 1 - \frac{50}{169} = \frac{119}{169}$

(c) Let $\theta = \sin^{-1}\frac{1}{3}$ and $\beta = \sin^{-1}\frac{2}{3} \Rightarrow \sin\theta = \frac{1}{3}$ and $\sin\beta = \frac{2}{3}$.

Now $\sin(\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3}) = \sin(\theta+\beta) = \sin\theta\cos\beta + \cos\theta\sin\beta$

$= \frac{1}{3} \times \frac{\sqrt{5}}{3} + \frac{2\sqrt{2}}{3} \times \frac{2}{3} = \dfrac{\sqrt{5}+4\sqrt{2}}{9}$

(d) Let $\theta = \sin^{-1}\frac{3}{4}$ and $\beta = \cos^{-1}\frac{1}{4} \Rightarrow \sin\theta = \frac{3}{4}$ and $\cos\beta = \frac{1}{4}$.
Now $\cos(\sin^{-1}\frac{3}{4} + \cos^{-1}\frac{1}{4}) = \cos(\theta+\beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$

$= \frac{\sqrt{7}}{4} \times \frac{1}{4} - \frac{3}{4} \times \frac{\sqrt{15}}{4} = \dfrac{\sqrt{7}-3\sqrt{15}}{16}$

6. (a) $-1 \le 1-x \le 1 \Rightarrow -2 \le -x \le 0 \Rightarrow 0 \le x \le 2$
(b) $-1 \le x^2 \le 1 \Rightarrow 0 \le x^2 \le 1 \Rightarrow |x| \le 1 \Rightarrow -1 \le x \le 1$
(c) $-1 \le 1-x^2 \le 1 \Rightarrow -2 \le -x^2 \le 0 \Rightarrow 0 \le x^2 \le 2 \Rightarrow |x| \le \sqrt{2}$
$\Rightarrow -\sqrt{2} \le x \le \sqrt{2}$

6. (d) $-1 \le -x^2 \le 1 \Rightarrow -1 \le x^2 \le 1 \Rightarrow 0 \le x^2 \le 1 \Rightarrow |x| \le 1 \Rightarrow -1 \le x \le 1$

(e) $-1 \le x^2 - 4 \le 1 \Rightarrow 3 \le x^2 \le 5 \Rightarrow -\sqrt{5} \le x \le -\sqrt{3}$ or $\sqrt{3} \le x \le \sqrt{5}$

(f) $-1 \le \sqrt{x-1} \le 1 \Rightarrow 0 \le x-1 \le 1 \Rightarrow 1 \le x \le 2$

7. (a) Let $t = \frac{3}{x-2}$. As $x \to 2^-$, $x - 2 \to 0^-$, and $t \to -\infty$.
Therefore $\lim_{x \to 2^-} \tan^{-1}\frac{3}{x-2} = \lim_{t \to -\infty} \tan^{-1} t = -\frac{\pi}{2}$.

(b) Let $t = x^2$. As $x \to \infty$, $x^2 \to \infty$, and $t \to \infty$.
Therefore $\lim_{x \to \infty} \tan^{-1}(x^2) = \lim_{t \to \infty} \tan^{-1} t = \frac{\pi}{2}$.

(c) Let $t = x - x^2$. As $x \to \infty$, $x - x^2 = x(1-x) \to -\infty$ and $t \to -\infty$.
Thererfore $\lim_{x \to \infty} \tan^{-1}(x - x^2) = \lim_{t \to -\infty} \tan^{-1} t = -\frac{\pi}{2}$.

(d) Let $t = \frac{x}{3-x}$. As $x \to 3^+$, $3 - x \to 0^-$, $\frac{x}{3-x} \to -\infty$ and $t \to -\infty$.
Therefore $\lim_{x \to 3^+} \tan^{-1}\left(\frac{x}{3-x}\right) = \lim_{t \to -\infty} \tan^{-1} t = -\frac{\pi}{2}$

8. (a) Let $\tan^{-1}\frac{3}{5} = A$, $\sin^{-1}\frac{3}{5} = B$ and $\tan^{-1}\frac{27}{11} = C$. To prove $A + B = C$ it is only necessary to prove $\tan(A+B) = \tan C$. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \dfrac{\frac{3}{5} + \frac{3}{4}}{1 - \frac{3}{5} \times \frac{3}{4}} = \frac{27}{20} \times \frac{20}{11} = \frac{27}{11} = \tan C$.

(b) Let $\tan^{-1}\frac{1}{7} = A$, $\tan^{-1}\frac{1}{8} = B$, $\tan^{-1}\frac{1}{18} = C$, and $\cot^{-1} 3 = D$.
To prove $A + B + C = D$ it is only necessary to prove that $\tan(A+B+C) = \tan D$.

$\tan(A+B) = \dfrac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \frac{1}{8}} = \frac{15}{55} = \frac{3}{11}$. $\tan[(A+B)+C] = \dfrac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{11} \times \frac{1}{18}} = \frac{65}{195} = \frac{1}{3} = \tan D$.

(c) Let $\theta = \sin^{-1} x$. Now $\sin(2\sin^{-1} x) = \sin 2\theta$

$= 2\sin\theta\cos\theta = 2x \times \dfrac{\sqrt{1-x^2}}{1} = 2x\sqrt{1-x^2}$.

(d) Let $\tan^{-1} m = A$, $\tan^{-1} n = B$ and $\cos^{-1}\dfrac{1-mn}{\sqrt{(1+m^2)(1+n^2)}} = C$.

To prove $A+B=C$ it is only necessary to prove $\cos(A+B)=\cos C$.

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$= \frac{1}{\sqrt{1+m^2}} \times \frac{1}{\sqrt{1+n^2}} - \frac{m}{\sqrt{1+m^2}} \times \frac{n}{\sqrt{1+n^2}}$$
$$= \frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}} = \cos C.$$

9. Let $\tan^{-1} a = A$, $\tan^{-1} c = C$, $\tan^{-1}\dfrac{a-b}{1+ab} = R$, and $\tan^{-1}\dfrac{b-c}{1+bc} = S$. To prove

$A-C=R+S$ it is only necessary to prove $\tan(A-C)=\tan(R+S)$.

$$\tan(R+S) = \frac{\dfrac{a-b}{1+ab} + \dfrac{b-c}{1+bc}}{1 - \dfrac{a-b}{1+ab} \times \dfrac{b-c}{1+bc}}$$

$$= \frac{(a-b)(1+bc)+(b-c)(1+ab)}{(1+ab)(1+bc)-(a-b)(b-c)}$$

$$= \frac{a+abc-b-b^2c+b+ab^2-c-abc}{1+bc+ab+ab^2c-ab+ac+b^2-bc}$$

$$= \frac{a-c-b^2c+ab^2}{1+ac+b^2+ab^2c} = \frac{(a-c)+b^2(a-c)}{(1+b^2)+ac(1+b^2)}$$

$$= \frac{(a-c)(1+b^2)}{(1+b^2)(1+ac)} = \frac{a-c}{1+ac} = \tan(A-C)$$

10. Let $\tan^{-1} x = A$, $\tan^{-1} y = B$, and $\tan^{-1} z = C$. Therefore $A+B+C=\pi$ and

$$\tan(A+B+C)=0 \Rightarrow \tan[(A+B)+C]=0 \Rightarrow \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}=0$$

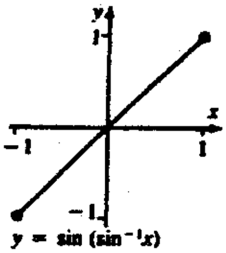
$$\Rightarrow \tan(A+B)+\tan C=0 \Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}+\tan C=0$$

$$\Rightarrow \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B}=0 \Rightarrow \text{numerator}=0.$$

Therefore $x+y+z-xyz=0$ and $x+y+z=xyz$.

Exercise 7.5

11. (a) $f(x)=\sin(\sin^{-1}x) \Rightarrow f(x)=x$ in the interval $[-1,1]$



$y = \sin(\sin^{-1}x)$

(b) $f(x)=\sin^{-1}(\sin x)$

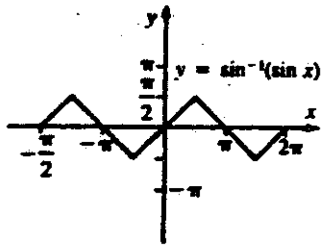
⋮

$\Rightarrow f(x)=-x,\ -\frac{3\pi}{2}\le x\le -\frac{\pi}{2}$

$\Rightarrow f(x)=x,\ -\frac{\pi}{2}\le x\le \frac{\pi}{2}$

$\Rightarrow f(x)=-x,\ \frac{\pi}{2}\le x\le \frac{3\pi}{2}$

⋮



EXERCISE 7.6

1. (a) $y' = \frac{1}{\sqrt{1-(x+1)^2}}\frac{d}{dx}(x+1) = \frac{1}{\sqrt{-x^2-2x}}$

(b) $y' = \frac{-1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}(x^2) = \frac{-2x}{\sqrt{1-x^4}}$ (c) $y' = \frac{1}{1+(3x)^2}\frac{d}{dx}(3x) = \frac{3}{1+9x^2}$

(d) $y' = 2\sin^{-1}x\frac{d}{dx}(\sin^{-1}x) = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$ (e) $y' = \frac{-1}{\sqrt{1-\left(\frac{x^6}{4}\right)}}\left(\frac{3x^2}{2}\right) = \frac{-3x^2}{\sqrt{4-x^6}}$

(f) $y' = (1+x^2)\left(\frac{1}{1+x^2}\right) + \tan^{-1}x\,(2x) = 1 + 2x\tan^{-1}x$

(g) $y' = \frac{-1}{\sqrt{1-(2x-1)}}\frac{1}{2}(2x-1)^{-\frac{1}{2}}(2) = \frac{-1}{\sqrt{(2-2x)(2x-1)}}$

(h) $y' = \frac{1}{1+\sin^2 x}\cos x = \frac{\cos x}{2-\cos^2 x}$

(i) $y' = \frac{1}{\sqrt{1-\frac{\cos^2 x}{(1+\sin x)^2}}} \times \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$

$= \frac{1+\sin x}{\sqrt{1+2\sin x+\sin^2 x-\cos^2 x}} \times \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$

$= \frac{-\sin x - 1}{(1+\sin x)\sqrt{2\sin^2 x + 2\sin x}} = \frac{-1}{\sqrt{2\sin x(\sin x+1)}}$

(j) $y' = \frac{\cos^{-1}x\frac{1}{\sqrt{1-x^2}} - \sin^{-1}x\frac{-1}{\sqrt{1-x^2}}}{(\cos^{-1}x)^2} = \frac{\cos^{-1}x - \sin^{-1}x}{(\cos^{-1}x)^2\sqrt{1-x^2}}$

(k) $y' = -(\tan^{-1}x)^{-2}\frac{1}{1+x^2} = \frac{-1}{(1+x^2)(\tan^{-1}x)^2}$

(l) $y' = -2(\cos^{-1}x^2)^{-3}\frac{-1}{\sqrt{1-x^4}}(2x) = \frac{4x}{(\cos^{-1}x^2)^3\sqrt{1-x^4}}$

(m) $y' = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times \frac{-1}{x^2} = \frac{1}{1+x^2} + \frac{x^2}{x^2+1} \times \frac{-1}{x^2} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$

(n) $y' = \frac{x\frac{-2x}{2\sqrt{1-x^2}} - \sqrt{1-x^2}}{x^2} + \frac{1}{\sqrt{1-x^2}} = \frac{-x^2-1+x^2+x^2}{x^2\sqrt{1-x^2}} = \frac{x^2-1}{x^2\sqrt{1-x^2}}$

(o) $y' = \frac{\sqrt{1-x^2} - x\frac{-2x}{2\sqrt{1-x^2}}}{1-x^2} - \frac{1}{\sqrt{1-x^2}} = \frac{1-x^2+x^2-1+x^2}{(1-x^2)^{\frac{3}{2}}} = \frac{x^2}{\sqrt{(1-x^2)^3}}$

(p) $y' = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-(1-x^2)}} \times \frac{-2x}{2\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + \frac{x}{|x|\sqrt{1-x^2}} = \frac{|x|+x}{|x|\sqrt{1-x^2}}$

(q) $y' = \cos(\sin^{-1}x^2)\frac{1}{\sqrt{1-x^4}}(2x) = \frac{2x\cos(\sin^{-1}x^2)}{\sqrt{1-x^4}}$

(r) $y' = \frac{1}{\sqrt{1-(\tan^{-1}x)^2}} \times \frac{1}{1+x^2} = \frac{1}{(1+x^2)\sqrt{1-(\tan^{-1}x)^2}}$

(s) $y' = x^2\frac{-\frac{-2}{x^2}}{\sqrt{1-\frac{4}{x^2}}} + \cos^{-1}\left(\frac{2}{x}\right)(2x) = \frac{2|x|}{\sqrt{x^2-4}} + 2x\cos^{-1}\left(\frac{2}{x}\right)$

2. $f'(x) = \frac{x}{1+x^2} + \tan^{-1}x$. $f'(1) = \frac{1}{2} + \tan^{-1}1 = \frac{1}{2} + \frac{\pi}{4} = \frac{2+\pi}{4}$ = slope of the tangent line.

3. $f'(x) = \dfrac{x}{4\sqrt{1-\frac{x^2}{16}}} + \sin^{-1}\left(\frac{x}{4}\right) + \dfrac{-x}{\sqrt{16-x^2}}$. $f'(2) = \dfrac{2}{4\sqrt{1-\frac{1}{4}}} + \sin^{-1}\frac{1}{2} - \dfrac{2}{\sqrt{12}}$

$= \frac{1}{\sqrt{3}} + \frac{\pi}{6} - \frac{1}{\sqrt{3}} = \frac{\pi}{6} =$ slope of the tangent line. $f(2) = 2\sin^{-1}\frac{1}{2} + \sqrt{12} = \frac{\pi}{3} + 2\sqrt{3}$.

The equation of the tangent is $y - \frac{\pi}{3} - 2\sqrt{3} = \frac{\pi}{6}(x-2)$
$\Rightarrow 6y - 2\pi - 12\sqrt{3} = \pi x - 2\pi \Rightarrow \pi x - 6y + 12\sqrt{3} = 0$.

4. $f'(x) = 4(3\tan^{-1}x)^3 \dfrac{3}{1+x^2}$. $f'(\sqrt{3}) = 4(3\tan^{-1}\sqrt{3})^3 \dfrac{3}{1+3} = 4\left(\frac{3\pi}{3}\right)^3\left(\frac{3}{4}\right) = 3\pi^3$.

5. $y^2\cos x + \sin x(2yy') = \dfrac{1}{1+x^2} - y' \Rightarrow (2y\sin x + 1)y' = \dfrac{1 - y^2\cos x - x^2y^2\cos x}{1+x^2}$

$\Rightarrow y' = \dfrac{1 - y^2\cos x - x^2y^2\cos x}{(1+x^2)(1+2y\sin x)}$

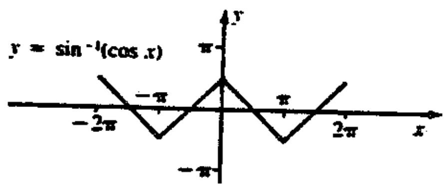
6. $f'(x) = \dfrac{(x-3)(6-2x)}{2\sqrt{6x-x^2}} + \sqrt{6x-x^2} + 9\dfrac{\frac{x}{3}}{\sqrt{1-\frac{(x-3)^2}{9}}}$

$= \dfrac{(x-3)(3-x)}{\sqrt{6x-x^2}} + \sqrt{6x-x^2} + \dfrac{9x}{\sqrt{-x^2+6x}}$.

$f'(3) = 0 + \sqrt{18-9} + \dfrac{27}{\sqrt{-9+18}} = 3 + 9 = 12$.

7. (a) $y' = \dfrac{-\sin x}{\sqrt{1-\cos^2 x}} = \dfrac{-\sin x}{\sqrt{\sin^2 x}} = \dfrac{-\sin x}{|\sin x|}$. Therefore $y' = -1$ if $\sin x > 0$ or

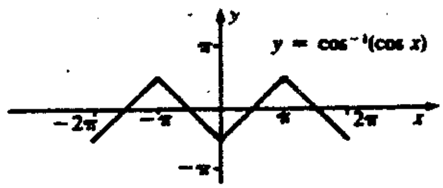
$y' = 1$ if $\sin x < 0$



Exercise 7.6

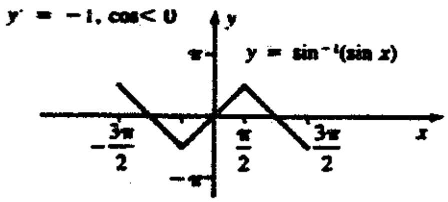
(b) $y' = \frac{-(-\sin x)}{\sqrt{1-\cos^2 x}} = \frac{\sin x}{\sqrt{\sin^2 x}} = \frac{\sin x}{|\sin x|}$. Therefore $y' = 1$ if $\sin x > 0$ or

$y' = -1$ if $\sin x < 0$.



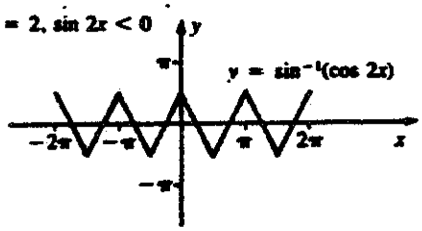
(c) $y' = \frac{\cos x}{\sqrt{1-\sin^2 x}} = \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{|\cos x|}$. Therefore $y' = 1$ if $\cos x > 0$ or

$y' = -1$ if $\cos x < 0$.



(d) $y' = \frac{-2\sin 2x}{\sqrt{1-\cos^2 2x}} = \frac{-2\sin 2x}{\sqrt{\sin^2 2x}} = \frac{-2\sin 2x}{|\sin 2x|}$. Therefore $y' = -2$ if $\sin 2x > 0$ or

$y' = 2$ if $\sin 2x < 0$.



REVIEW EXERCISE 7.7

1. (a) $\frac{1}{2}\lim_{\frac{1}{2}x\to 0}\frac{\sin\frac{1}{2}x}{\frac{1}{2}x}=\frac{1}{2}(1)=\frac{1}{2}$ (b) $\lim_{x\to 0}\frac{\sin x}{x}=1$

(c) $\frac{3}{5}\lim_{3x\to 0}\frac{\sin 3x}{3x}=\frac{3}{5}(1)=\frac{3}{5}$ (d) $\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\sin x}{\cos x}=(1)(0)=0$

(e) $\lim_{x\to 0}\frac{x}{\sin x}=1$ (f) $\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\cos x=(1)(1)=1$

(g) $\lim_{x\to 0}\frac{1-(1-2\sin^2 x)}{x}=2\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\sin x=2(1)(0)=0$

(h) $\lim_{x\to 0}\frac{2\sin^2 x}{x^2\cos^2 x}=2\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos^2 x}=2(1)^2(\frac{1}{1^2})=2$

(i) $\lim_{\frac{\pi}{2}-x\to 0}\frac{\frac{\pi}{2}-x}{\sin(\frac{\pi}{2}-x)}=1$

2. (a) $y'=4\tan^3 3x(\sec^2 3x)(3)=12\tan^3 3x\sec^2 3x$

(b) $y'=\frac{(1-2\cos x)\cos x-\sin x(2\sin x)}{(1-2\cos x)^2}=\frac{\cos x-2(\cos^2 x+\sin^2 x)}{(1-2\cos x)^2}=\frac{\cos x-2}{(1-2\cos x)^2}$

(c) $y'=\sec x^2\tan x^2(2x)=2x\sec x^2\tan x^2$

(d) $y'=\frac{(1+x^2)2\cot 2x(-\csc^2 2x)2-\cot^2 2x(2x)}{(1+x^2)^2}$

$=\frac{-2\cot 2x[2(1+x^2)\csc^2 2x+x\cot 2x]}{(1+x^2)^2}$

(e) $y'=-\csc(x^3+1)\cot(x^3+1)(3x^2)=-3x^2\csc(x^3+1)\cot(x^3+1)$

(f) $y'=2\sec\sqrt{x}\tan\sqrt{x}\,(\frac{1}{2\sqrt{x}})=\frac{\sec\sqrt{x}\tan\sqrt{x}}{\sqrt{x}}$

(g) $y'=\frac{1}{3}(x\tan x)^{-\frac{2}{3}}[x\sec^2 x+\tan x]=\frac{\tan x+x\sec^2 x}{3\sqrt[3]{x^2\tan^2 x}}$

(h) $y' = 2\cos(\tan x)[-\sin(\tan x)]\sec^2 x = -\sec^2 x \sin(2\tan x)$

(i) $y' = \dfrac{-1}{\sin^2(x-\sin x)}[\cos(x-\sin x)](1-\cos x) = \dfrac{(\cos x - 1)[\cos(x-\sin x)]}{\sin^2(x-\sin x)}$

3. (a) $y' = -\sin(x-y)(1-y') \Rightarrow [1-\sin(x-y)]y' = -\sin(x-y)$
$\Rightarrow y' = \dfrac{-\sin(x-y)}{1-\sin(x-y)}$

(b) $(1+y')\cos(x+y)+(1-y')\cos(x-y) = 0$
$\Rightarrow [\cos(x+y)-\cos(x-y)]y' = -\cos(x-y)-\cos(x+y)$

$\Rightarrow y' = \dfrac{\cos(x-y)+\cos(x+y)}{\cos(x-y)-\cos(x+y)}$

(c) $y' = (1+y')\sec^2(x+y) \Rightarrow [1-\sec^2(x+y)]y' = \sec^2(x+y)$

$\Rightarrow y' = \dfrac{\sec^2(x+y)}{1-\sec^2(x+y)} = \dfrac{\sec^2(x+y)}{-\tan^2(x+y)} = -\dfrac{\sec^2(x+y)}{y^2}$

(d) $-(1+y')\sin(x+y) = y\cos x + y'\sin x$
$\Rightarrow -[\sin(x+y)+\sin x]y' = \sin(x+y)+y\cos x$

$\Rightarrow y' = -\dfrac{y\cos x + \sin(x+y)}{\sin x + \sin(x+y)}$

(e) $-(xy'+y)\csc^2 xy + xy' + y = 0$
$\Rightarrow (-x\csc^2 xy + x)y' = y\csc^2 xy - y$

$\Rightarrow y' = \dfrac{y(\csc^2 xy - 1)}{x(1-\csc^2 xy)} = -\dfrac{y}{x}$

(f) $-(1-y')\csc(x-y)\cot(x-y)+(1+y')\sec(x+y)\tan(x+y) = 1$
$\Rightarrow [\csc(x-y)\cot(x-y)+\sec(x+y)\tan(x+y)]y'$
$= 1+\csc(x-y)\cot(x-y)-\sec(x+y)\tan(x+y)$

$\Rightarrow y' = \dfrac{1+\csc(x-y)\cot(x-y)-\sec(x+y)\tan(x+y)}{\csc(x-y)\cot(x-y)+\sec(x+y)\tan(x+y)}$

4. $y' = \cos x - \sin x$. The tangent line is horizontal when $y' = 0 \Rightarrow \sin x = \cos x$
$\Rightarrow \tan x = 1$, $0 \le x \le 2\pi \Rightarrow x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$. The points are $(\frac{\pi}{4}, \frac{2}{\sqrt{2}})$
and $(\frac{5\pi}{4}, -\frac{2}{\sqrt{2}})$.

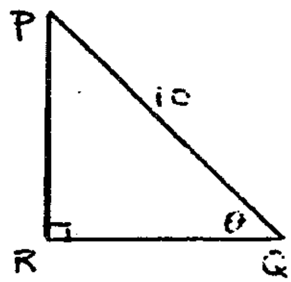
5. $y' = \sec^2 x$. When $x = \frac{\pi}{3}$, $y = \sqrt{3}$ and $y' = 4$ = the slope of the tangent line.
The equation of the tangent line is $y - \sqrt{3} = 4(x - \frac{\pi}{3})$
$\Rightarrow 12x - 3y + 3\sqrt{3} - 4\pi = 0$

6. $x\sec^2 y(y') + \tan y = y' \Rightarrow (x\sec^2 y - 1)y' = -\tan y \Rightarrow y' = \frac{\tan y}{1 - x\sec^2 y}$.

When $y = \frac{\pi}{4}$, $x\tan\frac{\pi}{4} = \frac{\pi}{4} - 1 \Rightarrow x = \frac{\pi}{4} - 1$. Therefore $y' = \frac{\tan\frac{\pi}{4}}{1 - (\frac{\pi}{4} - 1)\sec^2\frac{\pi}{4}}$

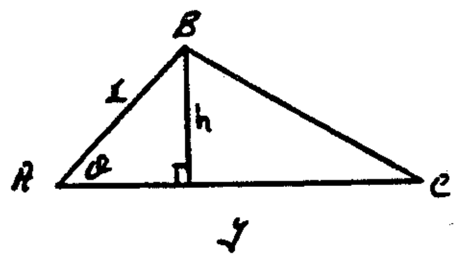
$= \frac{1}{1 - (\frac{\pi}{4} - 1)(\sqrt{2})^2} = \frac{1}{3 - \frac{\pi}{2}} = \frac{2}{6 - \pi}$ = slope of the tangent line.

7. Let the area of the triangle be A cm^2
and $\angle Q = \theta$ rad.
Since $A = \frac{1}{2}PR \times QR$,
$A = \frac{1}{2}(10\sin\theta)(10\cos\theta) = 25\sin 2\theta$.
$\frac{dA}{dt} = 50\cos 2\theta\frac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$ and $\frac{d\theta}{dt} = -\frac{\pi}{36}$,



$\frac{dA}{dt} = 50 \times \frac{1}{2} \times -\frac{\pi}{36} = -\frac{25}{36}\pi$. Therefore the area is decreasing at $\frac{25}{36}\pi$ cm/s^2.

8. Let the area of the triangle be A cm^2,
side AB = x cm, side AC = y cm, and
$\angle A = \theta$ rad. Let the altitude drawn from
B be h cm.
Since $A = \frac{1}{2}yh$,
$A = \frac{1}{2}y(x\sin\theta) = \frac{1}{2}xy\sin\theta$ and

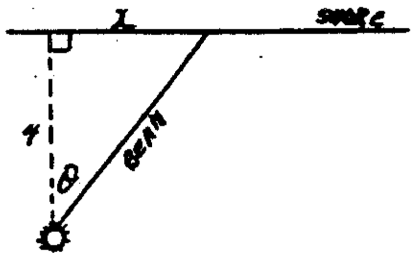


$\frac{dA}{dt} = \frac{1}{2}xy\cos\theta\frac{d\theta}{dt} + \frac{1}{2}\sin\theta\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$.

When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = \frac{1}{2}(40)(75)(\frac{\sqrt{3}}{2})(\frac{\pi}{180}) + \frac{1}{2}(\frac{1}{2})(120 + 150) = \frac{75\sqrt{3}}{19}\pi + \frac{135}{2}$

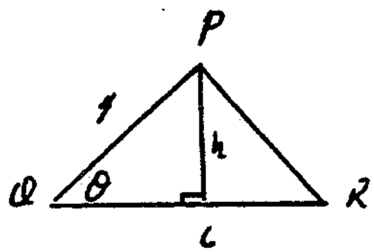
The area is increasing at approximately 90 cm^2/min.

9. Let the distance from the perpendicular that the beam moves along the shore be x km and the angle between the beam and the perpendicular be θ rad.
Since $x = 4\tan\theta$, $\frac{dx}{dt} = 4\sec^2\theta\frac{d\theta}{dt}$.



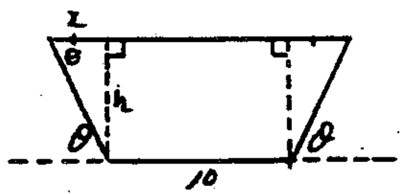
When $\theta = \frac{\pi}{4}$, $\frac{dx}{dt} = 4(\sqrt{2})^2(64\pi) = 512\pi$.
Therefore the beam sweeps along the shore at 512π km/min.

10. Let the area be $A\,cm^2$ and the angle between the given adjacent sides be θ.
h is the altitude drawn from P.
Since $A = 3h$ and $h = 4\sin\theta$
therefore $A = 12\sin\theta$ and $A' = 12\cos\theta$.
$A' = 0 \Rightarrow \cos\theta = 0,\ 0 \le \theta \le \pi \Rightarrow \theta = \frac{\pi}{2}$
$A'' = -12\sin\theta = -12 < 0$



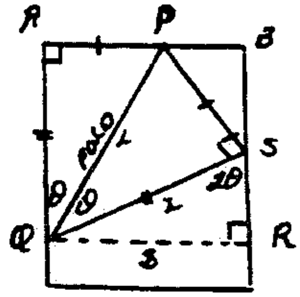
Therefore the triangle has maximum area when the enclosed angle is $\frac{\pi}{2} = 90°$.

11. The gutter will carry the maximum amount of water when the area of its cross-section (see diagram) is maximized. Let the area be A cm^2 and the distance between the parallel sides of the trapezoid be h cm. Let the distance from the perpendicular h to the corner of the trapezoid be x cm. The angle in the corner is θ, $0 < \theta < \frac{\pi}{2}$.



$A = \frac{1}{2}h[10 + (10 + 2x)] = 5\sin\theta(20 + 20\cos\theta) = 100\sin\theta + 50\sin 2\theta$.
$A' = 100\cos\theta + 100\cos 2\theta = 100\cos\theta + 200\cos^2\theta - 100$.
$A' = 0 \Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0 \Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0$
$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$.
$A'' = -100\sin\theta - 200\sin 2\theta = -\frac{100\sqrt{3}}{2} - \frac{200\sqrt{3}}{2} < 0 \Rightarrow$ maximum.
Therefore a bend of 60° maximizes the carrying capacity of the gutter.

12. In the diagram AP = PS and AQ = QS ie PS is AP in a different position and QS is AQ in a different position. ΔAPQ is congruent to ΔSQP and $\angle AQP = \angle PQS = \theta$. Therefore $\angle QSR = 2\theta$. Let PQ the length of the fold be L cm and QS be x cm.



Since $x = 8\csc 2\theta$ and $L = x\sec\theta$

therefore $L = 8\csc 2\theta \sec\theta = \dfrac{8}{2\sin\theta\cos^2\theta} = 4(\sin\theta\cos^2\theta)^{-1}$.

$L' = -4(\sin\theta\cos^2\theta)^{-2}(-2\sin^2\theta\cos\theta + \cos^3\theta)$.

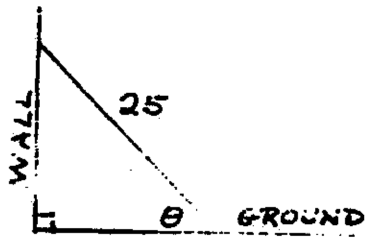
$L' = 0 \Rightarrow \cos^3\theta = 2\sin^2\theta\cos\theta \Rightarrow \tan^2\theta = \frac{1}{2} \Rightarrow \tan\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta \doteq 0.615$

If $\theta < 0.615$, $L' < 0$ and if $\theta > 0.615$, $L' > 0$ so a minimum occurs .

When $\tan\theta = \frac{1}{\sqrt{2}}$, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{\sqrt{2}}{1-\frac{1}{2}} = 2\sqrt{2}$ and $\csc 2\theta = \frac{3}{2\sqrt{2}}$.

Therefore $x = 8(\frac{3}{2\sqrt{2}}) = 6\sqrt{2}$ cm produces the minimum fold.

13. Let the angle between the beam and the ground be θ and the distance from the bottom of the beam to the wall be x m.



Since $x = 25\cos\theta$, $\dfrac{dx}{dt} = -25\sin\theta\dfrac{d\theta}{dt} \Rightarrow \dfrac{d\theta}{dt} = -\dfrac{3}{25\sin\theta}$.

When $x = 15$, $\sin\theta = \frac{20}{25}$ and $\dfrac{d\theta}{dt} = -\frac{3}{20}$.

Therefore the angle is decreasing at $\frac{3}{20}$ rad/min.

14. (a) A. Domain. The restricted domain $[0, 2\pi]$ is given.

B. Intercepts. $f(0) = 2(1) + (1)^2 = 3$ is the y-intercept. The x-intercepts occur when $\cos x(2 + \cos x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$.

C. Symmetry. None within our domain.

D. Asymptotes. None.

E. Intervals of Increase and Decrease. $y' = -2\sin x - 2\sin x\cos x$ $= -2\sin x(1 + \cos x)$. The critical numbers occur when $\sin x = 0$ or $\cos x = -1$ $\Rightarrow x = 0$ or $x = \pi$.

Interval	$-2\sin x$	$1 + \cos x$	y'	y
$0 < x < \pi$	$-$	$+$	$-$	decreasing
$\pi < x < 2\pi$	$+$	$+$	$+$	increasing

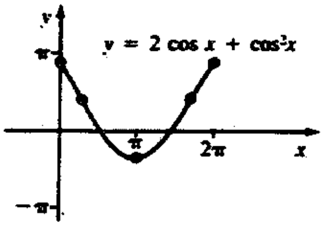
F. Local Maximum and Minimum Values. Since the curve changes from decreasing to increasing at $x=\pi$, $f(\pi)=2\cos\pi+\cos^2\pi=-2+1=-1$ is a local minimum. Examining the end points of the domain we get $f(0)=f(2\pi)=3$ is a maximum.

G. Concavity and Points of Inflection. $y''=-2\cos x-2\cos^2 x+2\sin^2 x$
$=-2\cos x-2\cos^2 x+2-2\cos^2 x=-2(2\cos^2 x+\cos x-1)$
$=-2(2\cos x-1)(\cos x+1)$. $y''=0 \Rightarrow \cos x=\frac{1}{2}$ or $\cos x=-1$
$\Rightarrow x=\frac{\pi}{3}, \pi$ or $\frac{5\pi}{3}$.

Interval	$-2(2\cos x-1)$	$\cos x+1$	y'	y
$0<x<\frac{\pi}{3}$	$-$	$+$	$+$	concave down
$\frac{\pi}{3}<x<\pi$	$+$	$+$	$+$	concave up
$\pi<x<\frac{5\pi}{3}$	$+$	$+$	$+$	concave up
$\frac{5\pi}{3}<x<2\pi$	$-$	$+$	$+$	concave down

Therefore points of inflection occur at $(\frac{\pi}{3}, 1.25)$ and $(\frac{5\pi}{3}, 1.25)$.

H. Sketch of the Graph.



14. (b) A. Domain. The restricted domain $[-\pi,\pi]$ is given.

B. Intercepts. $f(0)=0$ is the y-intercept. The x-intercepts occur when $\sin 2x=-x \Rightarrow x=0$.

C. Symmetry. $f(-x)=-x+\sin(-2x)=-x-\sin 2x=-f(x)$
$\Rightarrow$ symmetry about the origin.

D. Asymptotes. None.

E. Intervals of Increase and Decrease. $y'=1+2\cos 2x$. For critical numbers $y'=0 \Rightarrow \cos 2x=-\frac{1}{2}$, $-2\pi\le 2x\le 2\pi \Rightarrow 2x=-\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$
$\Rightarrow x=-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

Interval	$1+2\cos 2x$	y'
$-\pi<x<-\frac{2\pi}{3}$	$+$	increasing
$-\frac{2\pi}{3}<x<-\frac{\pi}{3}$	$-$	decreasing
$-\frac{\pi}{3}<x<\frac{\pi}{3}$	$+$	increasing
$\frac{\pi}{3}<x<\frac{2\pi}{3}$	$-$	decreasing
$\frac{2\pi}{3}<x<\pi$	$+$	increasing

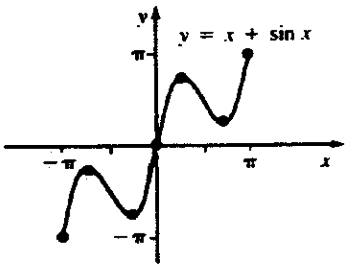
F. Local Maximum and Minimum Values. $f(-\frac{2\pi}{3}) \doteq -1.2$ and $f(\frac{\pi}{3}) \doteq 1.9$ are local maximums. $f(-\frac{\pi}{3}) \doteq -1.9$ and $f(\frac{2\pi}{3}) \doteq 1.2$ are local minimums. Examining the end points of the domain we get $f(-\pi) = -\pi$ an absolute minimum and $f(\pi) = \pi$ an absolute maximum.

G. Concavity and Points of Inflection. $y'' = -4\sin 2x$. $y'' = 0$ $\Rightarrow x = -\pi,\ -\frac{\pi}{2},\ 0,\ \frac{\pi}{2}$, or π.

Interval	$-4\sin 2x$	y
$-\pi < x < -\frac{\pi}{2}$	$-$	concave down
$-\frac{\pi}{2} < x < 0$	$+$	concave up
$0 < x < \frac{\pi}{2}$	$-$	concave down
$\frac{\pi}{2} < x < \pi$	$+$	concave up

The points of inflection are $(-\frac{\pi}{2}, -\frac{\pi}{2})$, $(0,0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$.

H. Sketch of the Graph.



15. $f(x) = \cos x + x$ and $f'(x) = -\sin x + 1$. A sketch of $y = \cos x$ and $y = -x$ gives the point of intersection at approximately -0.7. Let $x_1 = -0.7$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.7 - \frac{f(-0.7)}{f'(0.7)} = -0.739436$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -0.739085 \qquad x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -0.739085$$

Therefore the required point of intersection is $(-0.73908, 0.73908)$.

16. (a) Let $y = \cos^{-1}\frac{\sqrt{3}}{2} \Rightarrow \cos y = \frac{\sqrt{3}}{2},\ 0 \le y \le \pi \Rightarrow y = \frac{\pi}{6}$

(b) Let $y = \sin^{-1}\frac{1}{\sqrt{2}} \Rightarrow \sin y = \frac{1}{\sqrt{2}}.\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow y = \frac{\pi}{4}$

(c) Let $y = \tan^{-1}(-\sqrt{3}) \Rightarrow \tan y = -\sqrt{3}.\ -\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow y = -\frac{\pi}{3}$

(d) Let $y = \cos^{-1}\frac{1}{2} \Rightarrow \cos y = \frac{1}{2},\ 0 \le y \le \pi \Rightarrow \sin y = \frac{\sqrt{3}}{2}$

(e) Let $y = \tan^{-1}\frac{3}{4} \Rightarrow \tan y = \frac{3}{4},\ -\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow \cos y = \frac{4}{5}$

(f) Let $y = \sin^{-1}\frac{1}{\sqrt{5}} \Rightarrow \sin y = \frac{1}{\sqrt{5}},\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \tan y = \frac{1}{2}$

(g) Let $y = \tan^{-1} 1 \Rightarrow \tan y = 1,\ -\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow y = \frac{\pi}{4}$

(h) Let $y = \cos^{-1}\frac{\sqrt{3}}{2} \Rightarrow \cos y = \frac{\sqrt{3}}{2},\ 0 \le y \le \pi \Rightarrow y = \frac{\pi}{6}$

(i) Let $y = \sin^{-1}\frac{1}{2} \Rightarrow \sin y = \frac{1}{2},\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow y = \frac{\pi}{6}$

(j) Let $y = \sin^{-1}\frac{\sqrt{3}}{2} \Rightarrow \sin y = \frac{\sqrt{3}}{2},\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow y = \frac{\pi}{3}$

(k) Let $y = \sin^{-1}\frac{\sqrt{3}}{2} \Rightarrow \sin y = \frac{\sqrt{3}}{2},\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow y = \frac{\pi}{3}$

(l) Let $y = \cos^{-1}(-\frac{1}{2}) \Rightarrow \cos y = -\frac{1}{2},\ 0 \le y \le \pi \Rightarrow y = \frac{2\pi}{3}$

17. (a) $y' = \frac{1}{\sqrt{1-x^4}}(2x) = \frac{2x}{\sqrt{1-x^4}}$

(b) $y' = x^3 \frac{1}{\sqrt{1-\frac{x^2}{9}}}(\frac{1}{3}) + 3x^2\sin^{-1}(\frac{x}{3}) = \frac{x^3}{\sqrt{9-x^2}} + 3x^2\sin^{-1}(\frac{x}{3})$

(c) $y' = \frac{-1}{\sqrt{1-x}}(\frac{1}{2}x^{-\frac{1}{2}}) = \frac{-1}{2\sqrt{x(1-x)}}$ (d) $y' = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2}$

(e) $y' = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\left[\frac{(x+1)-(x-1)}{(x+1)^2}\right] = \frac{2|x+1|}{(x+1)^2\sqrt{(x+1)^2-(x-1)^2}}$

$= \frac{2|x+1|}{(x+1)^2\sqrt{4x}} = \frac{|x+1|}{(x+1)^2\sqrt{x}}$

(f) $y' = \frac{1}{1+\sin^4 x}(2\sin x\cos x) = \frac{\sin 2x}{1+\sin^4 x}$ (g) $y' = \frac{-1}{\sqrt{1-\frac{1}{x^2}}}\left(\frac{-1}{x^2}\right) = \frac{|x|}{x^2\sqrt{x^2-1}}$

(h) $y' = \frac{1}{\sqrt{1-\left(\sin^{-1}x\right)^2}}\frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{(1-x^2)[1-\left(\sin^{-1}x\right)^2]}}$

(i) $y' = \frac{1}{1+\left(\tan^{-1}x\right)^2}\frac{1}{1+x^2} = \frac{1}{(1+x^2)[1+\left(\tan^{-1}x\right)^2]}$

18. (a) Let $t = \frac{1}{x}$. As $x \to 0^-, t \to -\infty$. Therefore $\lim_{t\to-\infty} \tan^{-1} t = -\frac{\pi}{2}$

(b) Let $t = \frac{1}{\sin x}$. As $x \to 0^+$, $\sin x \to 0^+$ and $\frac{1}{\sin x} \to \infty$. Therefore $\lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$.

19. (a) $(-x\cos y)y' + \sin y + 3x^2 = \left(\frac{1}{1+y^2}\right)y'$

$\Rightarrow \left(-x\cos y - \frac{1}{1+y^2}\right)y' = -3x^2 - \sin y$

$\Rightarrow y' = \frac{(1+y^2)(3x^2+\sin y)}{x\cos y(1+y^2)+1}$

(b) $\frac{1}{\sqrt{1-x^2y^2}}(xy'+y) = \frac{-1}{\sqrt{1-(x+y)^2}}(1+y')$

$\Rightarrow \left(\frac{x}{\sqrt{1-x^2y^2}} + \frac{1}{\sqrt{1-(x+y)^2}}\right)y' = \frac{-y}{\sqrt{1-x^2y^2}} - \frac{1}{\sqrt{1-(x+y)^2}}$

$\Rightarrow y' = \frac{-y\sqrt{1-(x+y)^2} - \sqrt{1-x^2y^2}}{x\sqrt{1-(x+y)^2} + \sqrt{1-x^2y^2}}$

7.8 CHAPTER 7 TEST

1. (a) $\lim_{x \to 0} \frac{2\tan^2 x}{x^2} = 2\left[\lim_{x \to 0} \frac{\sin x}{x}\right]^2 \lim_{x \to 0} \frac{1}{\cos^2 x} = 2(1)^2 \times \frac{1}{(1)^2} = 2$

(b) $\lim_{x \to 0} \frac{1-\cos x}{x \sin x} = \lim_{x \to 0} \frac{1-\cos^2 x}{x \sin x(1+\cos x)}$

$= \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \frac{1}{1+\cos x} = (1)\left(\frac{1}{1+1}\right) = \frac{1}{2}$

2. (a) $y' = 2\sin(x^3-2)^{-4}\cos(x^3-2)^{-4}[-4(x^3-2)^{-5}(3x^2)]$

$= \frac{-24x^2 \sin(x^3-2)^{-4}\cos(x^3-2)^{-4}}{(x^3-2)^5}$

(b) $y' = \sec^2(\cos^3 x)[3\cos^2 x(-\sin x)] = -3\sin x \cos^2 x \sec^2(\cos^3 x)$

(c) $y' = \frac{(1+\cot x)(-\csc x \cot x) - \csc x(-\csc^2 x)}{(1+\cot x)^2}$

$= \frac{-\csc x(\cot x + \cot^2 x - \csc^2 x)}{(1+\cot x)^2} = \frac{-\csc x(\cot x - 1)}{(1+\cot x)^2}$

(d) $y' = \frac{(1+x^2)2\cot 2x(-2\csc^2 2x) - \cot^2 2x(2x)}{(1+x^2)^2}$

$= \frac{-2\cot 2x[2(1+x^2)\csc^2 2x + x\cot 2x]}{(1+x^2)^2}$

(e) $y' = 4\tan^3 x \sec^2 x - 4\sec^3 x \sec x \tan x = 4\sec^2 x \tan x(\tan^2 x - \sec^2 x)$
$= 4\sec^2 x \tan x(-1) = -4\sec^2 x \tan x$

(f) $y' = -2(\cos x \sin 2x)^{-3}[\cos x \cos 2x(2) + \sin 2x(-\sin x)]$

$= \frac{-2(2\cos x \cos 2x - 2\sin^2 x \cos x)}{(\cos x \sin 2x)^3} = \frac{4(\sin^2 x - \cos 2x)}{\cos^2 x \sin^3 2x} = \frac{4(3\sin^2 x - 1)}{\cos^2 x \sin^3 2x}$

3. $(-x\sin y)y' + \cos y - y\sin x + (\cos x)y' = xy' + y$
$\Rightarrow (-x\sin y + \cos x - x)y' = y - \cos y + y\sin x \Rightarrow y' = \dfrac{y - \cos y + y\sin x}{-x\sin y + \cos x - x}$

4. Let $y = \cos^{-1} -\frac{3}{5}$. Therefore $\cos y = -\frac{3}{5}$. $0 \le y \le \pi$. Now $\sin 2y = 2\sin y\cos y$

$= 2 \times \frac{4}{5} \times -\frac{3}{5} = -\frac{24}{25}$.

5. (a) $y' = \dfrac{1}{\sqrt{1 - \frac{x^2}{4}}}\left(\frac{1}{2}\right) = \dfrac{1}{2\sqrt{\frac{4-x^2}{4}}} = \dfrac{1}{\sqrt{4 - x^2}}$

(b) $y' = \dfrac{1}{1 + \left(\frac{x-1}{x+1}\right)^2}\left(\dfrac{x+1-(x-1)}{(x+1)^2}\right) = \dfrac{2}{(x+1)^2 + (x-1)^2} = \dfrac{2}{2(x^2+1)} = \dfrac{1}{x^2+1}$

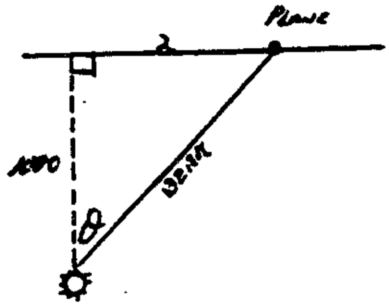
(c) $y' = \dfrac{-1}{\sqrt{1 - \sin^2 x}}(+\cos x) = \dfrac{-\cos x}{\sqrt{\cos^2 x}} = \dfrac{-\cos x}{|\cos x|} = \pm 1$

6. (a) $y' = 2\cos x - 2\sin 2x$. For critical numbers $y' = 0 \Rightarrow 2\cos x - 4\sin x\cos x = 0$
$\Rightarrow 2\cos x(1 - 2\sin x) = 0 \Rightarrow \cos x = 0$ and $x = -\frac{\pi}{2}, \frac{\pi}{2}$
or $\sin x = \frac{1}{2}$ and $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

(b)

Interval	$2\cos x$	$1 - 2\sin x$	y'	y
$-\pi < x < -\frac{\pi}{2}$	$-$	$+$	$-$	decreasing
$-\frac{\pi}{2} < x < \frac{\pi}{6}$	$+$	$+$	$+$	increasing
$\frac{\pi}{6} < x < \frac{\pi}{2}$	$+$	$-$	$-$	decreasing
$\frac{\pi}{2} < x < \frac{5\pi}{6}$	$-$	$-$	$+$	increasing
$\frac{5\pi}{6} < x < \pi$	$-$	$+$	$-$	decreasing

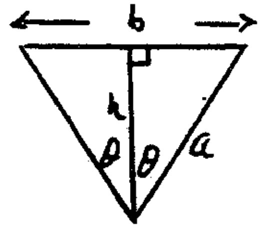
(c) $f(-\pi)=2(0)+1=1$. $f(-\frac{\pi}{2})=2(-1)+(-1)=-3$. $f(\frac{\pi}{6})=2(\frac{1}{2})+(\frac{1}{2})=\frac{3}{2}$.
$f(\frac{\pi}{2})=2(1)+(-1)=1$. $f(\frac{5\pi}{6})=2(\frac{1}{2})+(\frac{1}{2})=\frac{3}{2}$. $f(\pi)=2(0)+1=1$.
The absolute minimum in the given interval is -3
and the absolute maximum is 1.5.



7. Let the distance east of the searchlight be x m and the angle between the beam and the vertical be θ radians.
Since $\frac{x}{1000}=\tan\theta$, therefore $\frac{dx}{dt}=1000\sec^2\theta\frac{d\theta}{dt}$.

When $x=500$, $\sec\theta=\frac{\sqrt{5}}{2}$, and $-150=1000(\frac{5}{4})\frac{d\theta}{dt}$ $\Rightarrow$ $\frac{d\theta}{dt}=\frac{-600}{5000}$

Therefore the light is rotating counter-clockwise at $\frac{3}{25}$ rad/s



8. Let the side of the isosceles triangle be the constant a and half the vertical angle be θ. The volume is maximized when the area of the cross-section is maximized. Let the area be A. Since $A=\frac{1}{2}bh$, therefore $A=a\sin\theta\cos\theta=\frac{a}{2}\sin 2\theta$.

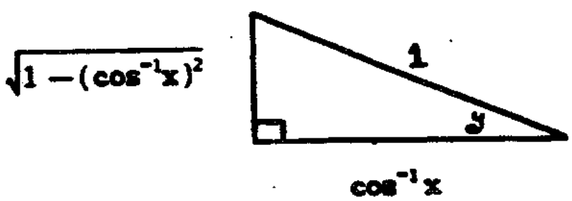
$\frac{dA}{d\theta}=\frac{a}{2}\cos 2\theta(2)=a\cos 2\theta.\frac{dA}{dt}=0$ $\Rightarrow$ $\cos 2\theta=0, 0<2\theta<\pi$ $\Rightarrow$ $2\theta=\frac{\pi}{2}$.

$\frac{d^2A}{d\theta^2}=-2a\sin 2\theta=-2a<0$ when $2\theta=\frac{\pi}{2}$.

Therefore a vertical angle of $\frac{\pi}{2}$ produces the maximum capacity.

9. $f(x)=\sin x+\cos x$ $\Rightarrow$ $f'(x)=\cos x-\sin x$ $\Rightarrow$ $f''(x)=-\sin x-\cos x$
$\Rightarrow$ $f'''=-\cos x+\sin x$ $\Rightarrow$ $f^{(4)}(x)=\sin x+\cos x$.
Therefore $f(x)=f^{(4)}(x)=f^{(8)}(x)=\cdots$ $\Rightarrow$ $n=4i$, $i\in N$.

10. $y'=\frac{-1}{\sqrt{1-(\cos^{-1}x)^2}}\times\frac{-1}{\sqrt{1-x^2}}$
$=\frac{1}{\sqrt{1-(\cos^{-1}x)^2}}\times\frac{1}{\sqrt{1-x^2}}=\frac{1}{|\sin y|\sqrt{1-x^2}}$



Cumulative Review For Chapters 4 to 7

1. (a) $\lim_{x\to -3^+} \frac{x}{x+3} = -\infty$ (b) $\lim_{x\to\infty} \frac{x}{x+3} = \lim_{x\to\infty} \frac{1}{1+\frac{3}{x}} = 1$

(c) $\lim_{x\to 1} \frac{x+2}{(x-1)^4} = \infty$ (d) $\lim_{x\to -\infty}(x^2+x^3) = \lim_{x\to -\infty} x^2(1+x) = -\infty$

(e) $\lim_{x\to\infty} \frac{2x^2+1}{x^2+3x+2} = \lim_{x\to\infty} \frac{2+\frac{1}{x^2}}{1+\frac{3}{x}+\frac{2}{x^2}} = 2$

(f) $\lim_{x\to -2^-} \frac{2x^2+1}{x^2+3x+2} = \lim_{x\to -2^-} \frac{2x^2+1}{(x+1)(x+2)} = \infty$

(g) $\lim_{x\to 0^-} \cot x = -\infty$ (h) $\lim_{x\to 0} \frac{\sin 5x}{8x} = \frac{5}{8} \lim_{5x\to 0} \frac{\sin 5x}{5x} = \frac{5}{8}$

(i) $\lim_{x\to 0} \frac{\tan 6x}{2x} = \lim_{x\to 0} \frac{\sin 6x}{2x\cos 6x} = 3 \lim_{6x\to 0} \frac{\sin 6x}{6x} \lim_{x\to 0} \frac{1}{\cos 6x} = 3(1)(1) = 3$

(j) $\lim_{x\to 0} \frac{\cos 2x - 1}{2x^2} = \lim_{x\to 0} \frac{-2\sin^2 x}{2x^2} = -\left[\lim_{x\to 0} \frac{\sin x}{x}\right]^2 = -1$

2. $\sin(x+y) = \frac{56}{65} \Rightarrow \sin x\cos y + \cos x \sin y = \frac{56}{65} \Rightarrow \frac{3}{5}\cos y + \frac{4}{5}\sin y = \frac{56}{65}$
$\Rightarrow$ (1) $3\cos y + 4\sin y = \frac{56}{13}$. Now (2) $3\cos y + 3\sin y = \frac{51}{13}$.
Subtracting (2) from (1) gives $\sin y = \frac{5}{13}$

3. (a) $\sin\frac{11\pi}{6} - \cos\frac{4\pi}{3} = \sin(2\pi - \frac{\pi}{6}) - \cos(\pi + \frac{\pi}{3}) = -\sin\frac{\pi}{6} + \cos\frac{\pi}{3} = -\frac{1}{2} + \frac{1}{2} = 0$

(b) $\frac{\tan\frac{7}{18}\pi + \tan\frac{1}{9}\pi}{1 - \tan\frac{7}{18}\pi\tan\frac{1}{9}\pi} = \tan\left(\frac{7}{18}\pi + \frac{1}{9}\pi\right) = \tan\frac{\pi}{2} = \text{undefined}$

(c) $\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

(d) $\cos^2\frac{\pi}{16} - \sin^2\frac{\pi}{16} = \cos\frac{\pi}{8}$. Now $\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1 \Rightarrow \frac{1}{\sqrt{2}} = 2\cos^2\frac{\pi}{8} - 1$

$\Rightarrow \cos^2\frac{\pi}{8} = \frac{1}{2\sqrt{2}} + \frac{1}{2} \Rightarrow \cos\frac{\pi}{8} = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$

(e) $\sin\frac{5}{8}\pi\cos\frac{5}{8}\pi = \frac{1}{2}\sin\frac{5}{4}\pi = \frac{1}{2}\sin\left(\pi + \frac{\pi}{4}\right) = -\frac{1}{2}\sin\frac{\pi}{4} = -\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2\sqrt{2}}$

(f) $\left[\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\right]^2 = \sin^2\frac{\pi}{8} - 2\cos\frac{\pi}{8}\sin\frac{\pi}{8} + \cos^2\frac{\pi}{8} = 1 - \sin\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$

4. (a) $\sin(a+b)\sin(a-b) + \cos(a+b)\cos(a-b)$

$= (\sin a \cos b + \cos a \sin b)(\sin a \cos b - \cos a \sin b)$

$\qquad + (\cos a \cos b - \sin a \sin b)(\cos a \cos b + \sin a \sin b)$

$= (\sin^2 a \cos^2 b - \cos^2 a \sin^2 b) + (\cos^2 a \cos^2 b - \sin^2 a \sin^2 b)$

$= \sin^2 a \cos^2 b - \cos^2 a \sin^2 b + \cos^2 b - \sin^2 a \cos^2 b - \sin^2 b + \cos^2 a \sin^2 b$

$= \cos^2 b - \sin^2 b = \cos 2b$

(b) $\sin 3a \csc a - \cos 3a \sec a = \dfrac{3\sin a - 4\sin^3 a}{\sin a} - \dfrac{4\cos^3 a - 3\cos a}{\cos a}$
$= 3 - 4\sin^2 a - 4\cos^2 a + 3 = 6 - 4(\sin^2 a + \cos^2 a) = 6 - 4 = 2$

(c) $\tan 4a = \dfrac{2\tan 2a}{1 - \tan^2 2a} = \dfrac{2\dfrac{2\tan a}{1-\tan^2 a}}{1 - \left[\dfrac{2\tan a}{1-\tan^2 a}\right]^2} = \dfrac{4\tan a(1-\tan^2 a)}{(1-\tan^2 a)^2 - 4\tan^2 a}$

$= \dfrac{4\tan a(1-\tan^2 a)}{1 - 2\tan^2 a + \tan^4 a - 4\tan^2 a} = \dfrac{4\tan a(1-\tan^2 a)}{1 - 6\tan^2 a + \tan^4 a}$

(d) $\dfrac{\sin(\pi - a)\cos\left(\frac{3\pi}{2} + a\right)}{\tan\left(\frac{\pi}{2} + a\right)\sin(-a)} = \dfrac{\sin a \sin a}{-\cot a(-\sin a)} = \dfrac{\sin a}{\cot a} = \sin a \tan a$

5. (a) $\sin x = \cos(2x - \pi) \Rightarrow \sin x = \cos(\pi - 2x) \Rightarrow \sin x = -\cos 2x$
$\Rightarrow \sin x = -1 + 2\sin^2 x \Rightarrow 2\sin^2 x - \sin x - 1 = 0$
$\Rightarrow (2\sin x + 1)(\sin x - 1) = 0 \Rightarrow \sin x = -\frac{1}{2}$ and $x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
or $\sin x = 1$ and $x = -\frac{3\pi}{2}, \frac{\pi}{2}$

(b) $2\cos^2 x \sin^2 x - \cos x \sin x = 0 \Rightarrow \frac{1}{2}\sin^2 2x - \frac{1}{2}\sin 2x = 0$
$\Rightarrow \sin 2x(\sin 2x - 1) = 0$
$\Rightarrow \sin 2x = 0$ and $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, $\sin 2x = 1$ and $x = \frac{\pi}{4}, \frac{5\pi}{4}$

(c) $\sin\frac{x}{2} + \cos\frac{x}{2} = \sqrt{2} \Rightarrow \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} = 2 \Rightarrow 1 + \sin x = 2$
$\Rightarrow \sin x = 1 \Rightarrow x = \frac{\pi}{2},$

(d) $\sin^2\left(x + \frac{\pi}{6}\right) - \cos^2\left(x + \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2} \Rightarrow -\cos\left(2x + \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$

$\Rightarrow \cos\left(2x + \frac{\pi}{3}\right) = -\frac{1}{\sqrt{2}} \Rightarrow 2x + \frac{\pi}{3} = \frac{3\pi}{4}$ and $x = \frac{5\pi}{24}$ or $2x + \frac{\pi}{3} = \frac{5\pi}{4}$ and $x = \frac{11\pi}{24}$

6. (a) Let $y = \tan^{-1}\frac{1}{\sqrt{3}} \Rightarrow \tan y = \frac{1}{\sqrt{3}},\ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow y = \frac{\pi}{6}$

(b) Let $y = \cos^{-1}\left(-\frac{1}{2}\right) \Rightarrow \cos y = -\frac{1}{2},\ 0 \le y \le \pi \Rightarrow y = \frac{2\pi}{3} \Rightarrow \tan\frac{2\pi}{3} = -\sqrt{3}.$

(c) $\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right] = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

7. (a) $y' = \frac{(\sin x + \cos x)(1) - x(\cos x - \sin x)}{(\sin x + \cos x)^2} = \frac{(1 + x)\sin x + (1 - x)\cos x}{(\sin x + \cos x)^2}$

(b) $y' = \frac{1}{2}(\cos^2 x - \sin^2 x)^{-\frac{1}{2}}(-2\cos x \sin x - 2\sin x \cos x)$

$= \frac{-2(2\sin x \cos x)}{2\sqrt{\cos^2 x - \sin^2 x}} = \frac{-\sin 2x}{\sqrt{\cos 2x}}$

(c) $y' = \frac{bx(a\cos ax) - \sin ax(b)}{b^2x^2} = \frac{ax\cos ax - \sin ax}{bx^2}$

(d) $y' = -\sin\left(\frac{2}{\sin x}\right)\left(-2\sin^{-2}x\cos x\right) = \frac{2\cos x}{\sin^2 x}\sin\left(\frac{2}{\sin x}\right) = 2\cot x\csc x\sin\left(\frac{2}{\sin x}\right)$

(e) $y' = \frac{(1 - \cot 2x)2\sec^2 2x - \tan 2x(2\csc^2 2x)}{(1 - \cot 2x)^2}$

$= \frac{2\sec^2 2x - 2\csc 2x\sec 2x - 2\csc 2x\sec 2x}{(1 - \cot 2x)^2} = \frac{2\sec 2x(\sec 2x - 2\csc 2x)}{(1 - \cot 2x)^2}$

(f) $y' = x^{-2}(-\csc x\cot x) + \csc x(-2x^{-3}) = -x^{-3}\csc x(x\cot x + 2)$

$= -\frac{\csc x(x\cot x + 2)}{x^3}$

(g) $(1 - y')\cos(x - y) = -y\sin x + y'\cos x$

$\Rightarrow [\cos(x - y) + \cos x]y' = y\sin x + \cos(x - y) \Rightarrow y' = \frac{y\sin x + \cos(x - y)}{\cos x + \cos(x - y)}$

(h) $2\tan x\sec^2 x = 2y'\sec^2 y\tan y \Rightarrow y' = \frac{\sec^2 x\tan x}{\sec^2 y\tan y}$

(i) $y' = \sec^2[\cos(\sin x)][-\sin(\sin x)]\cos x = -\cos x[\sin(\sin x)]\sec^2[\cos(\sin x)]$

(j) $y' = x^2\left[\frac{1}{\sqrt{1 - \frac{x^2}{4}}}\left(\frac{1}{2}\right)\right] + 2x\sin^{-1}\left(\frac{x}{2}\right) = \frac{x^2}{\sqrt{4 - x^2}} + 2x\sin^{-1}\left(\frac{x}{2}\right)$

(k) $y' = -\frac{1}{\sqrt{1 - \left(\frac{x}{x-1}\right)^2}}\left[\frac{(x - 1)(1) - x(1)}{(x - 1)^2}\right] = -\frac{x - 1 - x}{(x - 1)^2\sqrt{\frac{(x - 1)^2 - x^2}{(x - 1)^2}}}$

$$= \frac{|x-1|}{(x-1)^2\sqrt{1-2x}}$$

(l) $y' = \frac{1}{1+(\sqrt{x})^2}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{2\sqrt{x}\,(1+x)}$

8. (a) $f(x) = 4x^3 - 9x^2 + 6x - 1$; $f'(x) = 12x^2 - 18x + 6 = 6(2x-1)(x-1) = 0$ when $x = \frac{1}{2}$ or 1. $f'(x) > 0$ when $x < \frac{1}{2}$ and when $x > 1$. $f'(x) < 0$ when $\frac{1}{2} < x < 1$. So $f(\frac{1}{2}) = \frac{1}{4}$ is a local maximum and $f(1) = 0$ is a local minimum.

(b) $f(x) = \frac{x^3}{x^2-1}$; $f'(x) = \frac{3x^2(x^2-1)-2x^4}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2} = 0$ when $x = 0$ or $\pm\sqrt{3}$. $f'(x) > 0$ when $x < -\sqrt{3}$ and when $x > \sqrt{3}$. $f'(x) < 0$ when $-\sqrt{3} < x < \sqrt{3}$. So $f(-\sqrt{3}) = -\frac{3\sqrt{3}}{2}$ is a local maximum and $f(\sqrt{3}) = \frac{3\sqrt{3}}{2}$ is a local minimum.

(c) $f(x) = \sin x - \cos x \Rightarrow f'(x) = \cos x + \sin x$. For the critical numbers $\cos x + \sin x = 0 \Rightarrow \tan x = -1 \Rightarrow x = -\frac{\pi}{4}$. $f''(x) = -\sin x + \cos x$ and $f''(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} > 0$ Therefore $f(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$ is a local minimum.

(d) $f(x) = 2\sec x + \tan x \Rightarrow f'(x) = 2\sec x\tan x + \sec^2 x = \sec x(2\tan x + \sec x)$ $= \frac{2\sin x + 1}{\cos^2 x}$. For critical numbers $2\sin x + 1 = 0 \Rightarrow \sin x = -\frac{1}{2}$ or $\cos x = 0$. Therefore $x = -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$.

INTERVAL	f′(x)	f	INTERVAL	f′(x)	f
$(-\pi, -\frac{5\pi}{6})$	+	inc	$(\frac{\pi}{2}, \frac{7\pi}{6})$	+	inc
$(-\frac{5\pi}{6}, -\frac{\pi}{2})$	−	dec	$(\frac{7\pi}{6}, \frac{3\pi}{2})$	−	dec
$(-\frac{\pi}{2}, -\frac{\pi}{6})$	−	dec	$(\frac{3\pi}{2}, \frac{11\pi}{6})$	−	dec
$(-\frac{\pi}{6}, \frac{\pi}{2})$	+	inc	$(\frac{11\pi}{6}, 2\pi)$	+	inc

Therefore $f(-\frac{5\pi}{6}) = f(\frac{7\pi}{6}) = -\sqrt{3}$ are local maxima and $f(-\frac{\pi}{6}) = f(\frac{11\pi}{6}) = \sqrt{3}$ are local minima.

9. (a) $f(x) = x^3 - 6x^2 + 9x + 2$, $\frac{1}{2} \le x \le \frac{9}{2}$.
$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$ when $x = 1$ or 3. $f(1) = 6$, $f(3) = 2$, and checking the endpoints, $f(\frac{1}{2}) = \frac{41}{8}$ and $f(\frac{9}{2}) = \frac{97}{8}$. So the absolute minimum is $f(3) = 2$ and the absolute maximum is $f(\frac{9}{2}) = \frac{97}{8}$.

(b) $f'(x) = 2\sin x\cos x + 2\cos x = 2\cos x(\sin x + 1)$. Critical numbers occur when $\cos x = 0$ or $\sin x = -1 \Rightarrow x = \frac{\pi}{2}$. Since $f'(x) = \sin 2x + 2\cos x$, $f''(x) = 2\cos 2x - 2\sin x$ and $f''(\frac{\pi}{2}) = -2 - 2 < 0$. Therefore $f(\frac{\pi}{2}) = 1 + 2 = 3$ is a local maximum. Test the end-points of the domain: $f(0) = 0$ and $f(\pi) = 0$. Therefore $f(\frac{\pi}{2}) = 3$ is the absolute maximum and $f(0) = f(\pi) = 0$ is the absolute minimum.

10. $f(x) = \sin^2 x + 2\cos x \Rightarrow f'(x) = 2\sin x\cos x - 2\sin x$
$\Rightarrow f''(x) = -2\sin^2 x + 2\cos^2 x - 2\cos x = 2(-1 + \cos^2 x + \cos^2 x - \cos x)$
$= 2(2\cos^2 x - \cos x - 1) = 2(2\cos x + 1)(\cos x - 1)$.
$f''(x) = 0 \Rightarrow 2\cos^2 x - \cos x - 1 = 0 \Rightarrow (2\cos x + 1)(\cos x - 1) = 0$
$\Rightarrow \cos x = -\frac{1}{2}$ and $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ or $\cos x = 1$ and $x = 0, 2\pi$.

INTERVAL	$2\cos x + 1$	$\cos x - 1$	$f''(x)$	$f(x)$
$[0, \frac{2\pi}{3}]$	+	−	−	concave down
$[\frac{2\pi}{3}, \frac{4\pi}{3}]$	−	−	+	concave up
$[\frac{4\pi}{3}, 2\pi]$	+	−	−	concave down

11. (a) $y = 2x^2 - x^4$. A. Domain is R.

B. The y-intercept is 0 and the x-intercepts are 0 and $\pm\sqrt{2}$.

C. $f(-x) = f(x)$ so the function is even and is symmetric about the y-axis.

D. There are no asymptotes, but $\lim_{x \to \pm\infty} x^2(2 - x^2) = -\infty$.

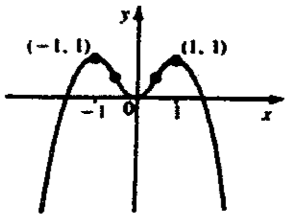
E. $f'(x) = 4x - 4x^3 = 4x(1 - x)(1 + x) = 0$ when $x = 0$ or ± 1.

Interval	x	$1 - x$	$1 + x$	f'	f
$(-\infty, -1)$	−	+	−	+	increasing
$(-1, 0)$	−	+	+	−	decreasing
$(0, 1)$	+	+	+	+	increasing
$(1, \infty)$	+	−	+	−	decreasing

F. $f(\pm 1) = 1$ are maximum values and $f(0) = 0$ is a minimum value.

G. $f''(x) = 4 - 12x^2 = 4(1 - 3x^2) = 0$ when $x = \pm\frac{1}{\sqrt{3}}$. f is CD on $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$. f is CU on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. There are inflection points at $(\pm\frac{1}{\sqrt{3}}, \frac{5}{9})$.

H.



(b) $y = x\sqrt{x+1}$. A. Domain is $[-1,\infty)$.

B. The y-intercept is 0 and the x-intercepts are 0 and -1.

C. There is no symmetry.

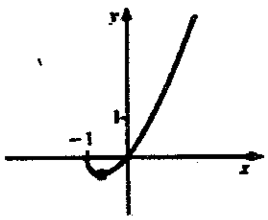
D. There are no asymptotes but $\lim_{x\to\infty} x\sqrt{x+1} = \infty$.

E. $f'(x) = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} = \frac{2x+2+x}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}} = 0$ when $x = -\frac{2}{3}$. f is decreasing on $(-1,-\frac{2}{3})$ and f is increasing on $(-\frac{2}{3},\infty)$.

F. $f(-\frac{2}{3}) = -\frac{2\sqrt{3}}{9}$ is an absolute minimum.

G. $f''(x) = \dfrac{6\sqrt{x+1} - \frac{3x+2}{\sqrt{x+1}}}{4(x+1)} = \dfrac{6x+6-3x-2}{4(x+1)^{\frac{3}{2}}} = \dfrac{3x+2}{4(x+1)^{\frac{3}{2}}}$. So f is CU on $(-1,\infty)$.

H.



(c) $y = \frac{x^2}{x^2-1}$. A. Domain is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

B. The y-intercept is 0 and the x-intercept is 0.

C. $f(-x) = f(x)$, so the the function is even and symmetric about the y-axis.

D. $\lim_{x\to\pm\infty} \frac{x^2}{x^2-1} = 1$, so $y = 1$ is a horizontal asymptote.

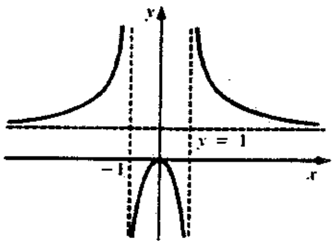
$\lim_{x\to 1^-} f(x) = \lim_{x\to -1^+} f(x) = -\infty$ and $\lim_{x\to -1^-} f(x) = \lim_{x\to 1^+} f(x) = \infty$, so $x = \pm 1$ are vertical asymptotes.

E. $f'(x) = \frac{2x(x^2-1)-2x^3}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2} = 0$ when $x = 0$. f is increasing on $(-\infty,-1)$ and $(-1,0)$. f is decreasing on $(0,1)$ and $(1,\infty)$.

F. $f(0) = 0$ is a local maximum.

G. $f''(x) = \frac{-2(x^2-1)^2+8x^2(x^2-1)}{(x^2-1)^4} = \frac{6x^2+2}{(x^2-1)^3}$. So y is CD on $(-1,1)$ and CU on $(-\infty,-1)$ and $(1,\infty)$. There are no inflection points.

H.



(d) A. Domain. The restricted domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is given.

B. Intercepts. The y-intercept is $f(0)=1$. The x-intercept occurs when $\cos 2x - x = 0 \Rightarrow x \doteq 0.51$.

C. Symmetry. None. $f(-x) = \cos 2x + x \neq f(x)$ or $-f(x)$.

D. Asymptotes. None.

E. Intervals of Increase or Decrease. $f(x) = \cos 2x - x$

$\Rightarrow f'(x) = -2\sin 2x - 1$.

For critical numbers $\sin 2x = -\frac{1}{2}$ $-\pi \le 2x \le \pi$

$\Rightarrow 2x = -\frac{5\pi}{6}, -\frac{\pi}{6} \Rightarrow x = -\frac{5\pi}{12}, -\frac{\pi}{12}$.

INTERVAL	f'(x)	f(x)
$\left(-\frac{\pi}{2}, -\frac{5\pi}{12}\right)$	−	decreasing
$\left(-\frac{5\pi}{12}, -\frac{\pi}{12}\right)$	+	increasing
$\left(-\frac{\pi}{12}, \frac{\pi}{2}\right)$	−	decreasing

F. Local Maximum and Minimum Values. $f(-\frac{5\pi}{12}) = \cos\left(-\frac{5\pi}{6}\right) + \frac{5\pi}{12} \doteq 0.44$ is a local minimum and $f(-\frac{\pi}{12}) = \cos\left(-\frac{\pi}{6}\right) + \frac{\pi}{12} \doteq 1.13$ is a local maximum.

G. Concavity and Points of Inflection. $f''(x) = -4\cos 2x$. $-4\cos 2x = 0$

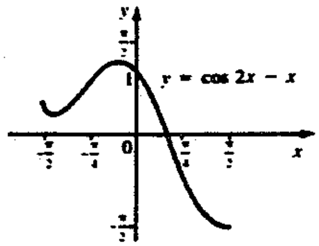
$\Rightarrow \cos 2x = 0, -\pi \le 2x \le \pi \Rightarrow 2x = -\frac{\pi}{2}, \frac{\pi}{2} \Rightarrow x = -\frac{\pi}{4}, \frac{\pi}{4}$.

INTERVAL	$f''(x)$	$f(x)$
$\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$	+	concave up
$\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$	−	concave down
$\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$	+	concave up

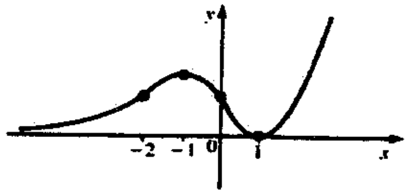
The points of inflection are

$\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, -\frac{\pi}{4}\right)$

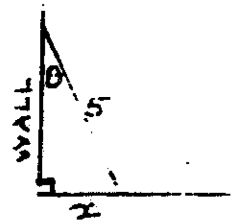
H. Sketch of the Graph.



12.

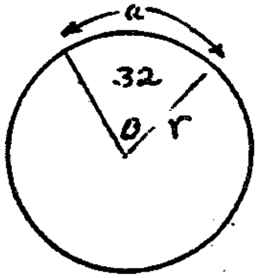


13. Let the angle between the wall and the ladder be θ and the distance from the foot of the ladder to the wall be x m. Therefore $\frac{x}{5} = \sin\theta \Rightarrow \frac{dx}{dt} = 5\cos\theta\frac{d\theta}{dt}$.
When $x = 3$, $\cos\theta = \frac{4}{5}$ and $\frac{1}{10} = 5\left(\frac{4}{5}\right)\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{1}{40}$.
The angle is increasing $\frac{1}{40}$ rad/s.



14. Let d represent the distance from a point (x,y) to the point (3,0).
$d^2 = (3-x)^2 + (0-y)^2 = (3-x)^2 + y^2$, but $y^2 = 2x$ from the equation of the parabola, so $f(x) = d^2 = (3-x)^2 + 2x = x^2 - 4x + 9$. Now, minimize d^2.
$f'(x) = 2x - 4 = 0$ when $x = 2$. $f'(x) < 0$ when $x < 2$ and $f'(x) > 0$ when $x > 2$, so f has a minimum at $x = 2$. Thus the points closest to (3,0) on $y^2 = 2x$ are $(2, \pm 2)$.

15. Let the arc length of the sector be a and the radius be r. The perimeter, $P = 2r + a = 2r + r\theta$.
Now $32 = \frac{1}{2}r^2\theta \Rightarrow \frac{64}{\theta} = r^2 \Rightarrow r = \frac{8}{\sqrt{\theta}}$.
Therefore $P = r(2+\theta) = \frac{8}{\sqrt{\theta}}(2+\theta) = 16\theta^{-\frac{1}{2}} + 8\theta^{\frac{1}{2}}$
$P' = -8\theta^{-\frac{3}{2}} + 4\theta^{-\frac{1}{2}} = 4\theta^{-\frac{3}{2}}(-2+\theta)$. $P' = 0 \Rightarrow \theta = 2$.
$P'' = 12\theta^{-\frac{5}{2}} - 2\theta^{-\frac{3}{2}} = \theta^{-\frac{5}{2}}(12 - 2\theta) > 0$ when $\theta = 2$.
Therefore $\theta = 2$rad produces the minimum perimeter.



16. (a) The demand function is $p(x) = 400 - \frac{20}{100}(x - 800) = 560 - \frac{1}{5}x$.

(b) The revenue function is $R(x) = x\,p(x) = 560x - \frac{1}{5}x^2$. $R'(x) = 560 - \frac{2}{5}x = 0$ when $x = 1400$. $p(1400) = 400 - \frac{1}{5}(600) = 280$. Thus the price that will maximize revenue is \$280.

REVIEW AND PREVIEW TO CHAPTER 8

EXERCISE 1.

1. (a) $(-3)^5 = -243$

   (b) $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

   (c) $2^{-3}5^4 = \frac{625}{8}$

   (d) $3^{-2} - (1.7)^0 = \frac{1}{9} - 1 = -\frac{8}{9}$

   (e) $36^{\frac{1}{2}} = 6$

   (f) $(-64)^{\frac{1}{3}} = -4$

   (g) $125^{\frac{2}{3}} = 25$

   (h) $9^{-\frac{7}{2}} = 3^{-7} = \frac{1}{2187}$

2. (a) $128 = 2^7$

   (b) $2^6 \times 8^4 = 2^6 \times (2^3)^4 = 2^6 \times 2^{12} = 2^{18}$

   (c) $(2^9)^4 = 2^{36}$

   (d) $\frac{1}{4} = 2^{-2}$

   (e) $\frac{2^{3.1}}{2^{4.6}} = 2^{-\frac{3}{2}}$

   (f) $\sqrt{2} = 2^{\frac{1}{2}}$

   (g) $4\sqrt{2} = 2^2 \times 2^{\frac{1}{2}} = 2^{\frac{5}{2}}$

   (h) $1 = 2^0$

3. (a) $(12x^2y^4)(\frac{1}{2}x^5y) = 6x^7y^5$

   (b) $(2s^3t^{-1})(\frac{1}{4}s^6)(16t^4) = 8s^9t^3$

   (c) $\frac{x^9(2x)^4}{x^3} = 16x^{10}$

   (d) $\frac{a^{-3}b^4}{a^{-5}b^5} = \frac{a^2}{b}$

   (e) $(rs)^3(2s)^{-2}(4r)^4 = 64r^7s$

   (f) $(2u^2v^3)^3(3u^3v)^{-2} = \frac{8v^7}{9}$

   (g) $\frac{(x^2y^3)^4(xy^4)^{-3}}{x^2y} = \frac{x^3}{y}$

   (h) $\left(\frac{c^4d^3}{cd^2}\right)\left(\frac{d^2}{c^3}\right)^3 = \frac{d^7}{c^6}$

   (i) $\frac{a^{-1}+b^{-1}}{(a+b)^{-1}} = (a+b)\left[\frac{1}{a}+\frac{1}{b}\right] = (a+b)\left[\frac{a+b}{ab}\right] = \frac{(a+b)^2}{ab}$

   (j) $\frac{(y^{10}z^{-5})^{\frac{1}{5}}}{(y^{-2}z^3)^{\frac{1}{3}}} = \frac{y^2z^{-1}}{y^{-\frac{2}{3}}z} = \frac{y^{\frac{8}{3}}}{z^2}$

   (k) $\frac{(9st)^{\frac{3}{2}}}{(27s^3t^{-4})^{\frac{2}{3}}} = \frac{27s^{\frac{3}{2}}t^{\frac{3}{2}}}{9s^2t^{-\frac{8}{3}}} = \frac{3t^{\frac{25}{6}}}{s^{\frac{1}{2}}}$